Taux de déclin au second ordre: résumé

RESUME

Taux de déclin PBA:

$$\xi_n = \frac{1}{n}\omega[f,f]$$
$$\xi_{u_i} = \frac{1}{nV_T}\omega[f, V_i f \ ]$$
$$\xi_T = \frac{m}{nk_BTd}\omega[f \ , V^2 f \ ] - \xi_n$$

Pour évaluer les contributions à l'ordre 2 qui contribuent au 1er ordre: doivent être linéaires dans les gradients car ce sont les seules qui contribuent à l'analyse de stabilité linéaire, i.e. $f^{(1)}f^{(1)} = O(\delta^2)$; $f^{(2)} = f_L^{(2)} + f^{(2)}_{nL}$, où $f_L^{(2)}$ est linéaire dans les gradients et $f_{nL}^{(2)}$ est non-linéaire et sera négligé. Ainsi:

$$\xi_{n,L}^{(2)} = \frac{2}{n}\omega[f^{(0)}, f_L^{(2)}]$$
$$\xi_{u_i,L}^{(2)} = \frac{1}{nV_T}\omega[f^{(0)}, V_i f_L^{(2)}] + \frac{1}{nV_T}\omega[f_L^{(2)}, V_i f^{(0)}]$$
$$\xi_{T,L}^{(2)} = \frac{m}{nk_BTd}\omega[f^{(0)}, V^2 f_L^{(2)}] + \frac{m}{nk_BTd}\omega[f_L^{(2)}, V^2 f^{(0)}] - \xi_n^{(2)}$$

Eq. de Chapman-Enskog pour $f^{(2)}$: collecte les termes d'ordre 2 dans l'Eq. de Boltzmann (46):

$$\partial_t^{(0)} f^{(2)} + \partial_t^{(1)} f^{(1)} + \partial_t^{(2)} f^{(0)} + V_1\cdot\nabla_1 f^{(1)} = \underbrace{p J_a[f^{(0)}, f^{(2)}] + p J_a[f^{(2)}, f^{(0)}] + (1-p) J_c[f^{(0)}, f^{(2)}] + (1-p) J_c[f^{(2)}, f^{(0)}]}_{:= -Jf^{(2)}} + O(\delta^3)$$

$$\Longrightarrow [\partial_t^{(0)} + J] f^{(2)} = -\partial_t^{(2)} f^{(0)} - (\partial_t^{(1)} + V_1\cdot\nabla) f^{(1)}$$

On développe avec les Eqs. de bilan, et $\xi^{(2)}$ étant des scalaires, il ne reste que les contributions sous trace et scalaires et on a pour $\delta f_L^{(2)}$:

$$\boxed{[\partial_t^{(0)} + J]\,\delta f_L^{(2)} - p\xi_{nL}^{(2)} f^{(0)} + p\,V_T\,\xi_{u_iL}^{(2)}\frac{\partial f^{(0)}}{\partial V_i} - pT\xi_{T,L}^{(2)}\frac{\partial f^{(0)}}{\partial T} = (K\nabla^2 T + \mu\nabla^2 n)\left[\frac{2m}{d(d+2)}\frac{\beta^3}{n} S_j(V) V_j \mathcal{M}(V) - \frac{2}{nk_Bd}\frac{\partial f^{(0)}}{\partial T}\right]}, \quad (1)$$

où $\delta f_L^{(2)}$ est la partie résiduelle de $f^{(2)}$ qui donne lieu à $\xi_L^{(2)}$, et $\xi_{u_iL}^{(2)}\partial f^{(0)}/\partial v_i = d\xi_u^{(2)}\partial f^{(0)}/\partial V$. La sol. $\delta f_L^{(2)}$ est de la forme:

$$\delta f_L^{(2)} = M(n,T,V)\nabla^2 T + N(n,T,V)\nabla^2 n,$$

que l'on substitue dans les taux de déclin à l'ordre 2 (on note $\xi_{A,L}^{(2)} = \xi_A^{(2)}$):

$$\xi_n^{(2)} = \xi_{n,1}^{(2)}\nabla^2 T + \xi_{n,2}^{(2)}\nabla^2 n$$
$$\xi_{u_i}^{(2)} = \xi_{u_i,1}^{(2)}\nabla^2 T + \xi_{u_i,2}^{(2)}\nabla^2 n$$
$$\xi_T^{(2)} = \xi_{T,1}^{(2)}\nabla^2 T + \xi_{T,2}^{(2)}\nabla^2 n$$

Pour évaluer (1) on doit calculer $\partial_t^{(0)}\delta f_L^{(2)}$ avec une analyse de dépendance fonctionnelle; on collecte les termes en $M(n,T,V)$ et $N(n,T,V)$, et on obtient 2 équations. On fait ensuite un développement en polynôme de Sonine:

$$M(V) = C_T^{(2)} S_2(V^2/V_T^2)\mathcal{M}(V) \quad ; \quad S_2(c^2) = 1/2\, c^4 - \frac{d+2}{2}c^2 + \frac{d(d+2)}{8} \quad ; \quad S_i(V) = \left(\frac{m}{2}V^2 - \frac{d+2}{2}k_BT\right)V_i$$
$$N(V) = C_n^{(2)} S_2(V^2/V_T^2)\mathcal{M}(V) \qquad V_T = \sqrt{2/\beta m}$$
$$\mathcal{M}(V) = 1/\pi^{d/2}\, n/V_T^d\, e^{-c^2}$$

Les taux de déclin ont alors la forme:

$$\xi_{n,1}^{(2)} = C_T^{(2)}\frac{2}{n}\omega[f^{(0)}, S_2\mathcal{M}]$$
$$\xi_{n,2}^{(2)} = C_n^{(2)}\frac{2}{n}\omega[f^{(0)}, S_2\mathcal{M}]$$
$$\xi_{u_i,1}^{(2)} = C_T^{(2)}\frac{1}{nV_T}\omega[f^{(0)}, V_i S_2\mathcal{M}] + C_T^{(2)}\frac{1}{nV_T}\omega[S_2\mathcal{M}, V_i f^{(0)}]$$
$$\xi_{u_i,2}^{(2)} = C_n^{(2)}\frac{1}{nV_T}\omega[f^{(0)}, V_i S_2\mathcal{M}] + C_n^{(2)}\frac{1}{nV_T}\omega[S_2\mathcal{M}, V_i f^{(0)}]$$
$$\xi_{T,1}^{(2)} = C_T^{(2)}\frac{m}{nk_BTd}\omega[f^{(0)}, V^2 S_2\mathcal{M}] + C_n^{(2)}\frac{1}{nk_BTd}\omega[S_2\mathcal{M}, V^2 f^{(0)}] - \xi_{n,1}^{(2)}$$
$$\xi_{T,2}^{(2)} = C_T^{(2)}\frac{m}{nk_BTd}\omega[f^{(0)}, V^2 S_2\mathcal{M}] + C_n^{(2)}\frac{1}{nk_BTd}\omega[S_2\mathcal{M}, V^2 f^{(0)}] - \xi_{n,2}^{(2)}$$

En utilisant la dépendance fonctionnelle $C_T^{(2)} \propto n^{-2}T^{-1}$, $C_n^{(2)} \propto n^{-3}T^0$, et en adimensionalisant on obtient (on utilise le fait que $\xi_{u_i}^{(2)} = 0$ pour les modèles de Maxwell et VMP, et que $C_n^{(2)} = 0$ aussi ; et $\phi = \phi_n\frac{4}{(d+2)(d+4)}$)

$$\boxed{\begin{aligned} \left(2p\xi_n^{(0)*} - p\tfrac{5}{2}\xi_T^{(0)*} + V_q^* - p\tfrac{2(d+2)}{d+4}\right) C_T^{(2)*} &= p\,\tfrac{1}{2}\,\xi_n^{(0)*} C_n^{(2)*} + \tfrac{8}{d(d+2)}K^* \\ \left(p\,\xi_n^{(0)*} + V_q^* - p\tfrac{2(d+2)}{d+4}\right) C_n^{(2)*} &= p\ \xi_T^{(0)*} C_T^{(2)} + \tfrac{8}{d(d+2)}\mu^* \end{aligned}}$$

$\Longrightarrow$ résolution pour $C_T^{(2)*}$ et $C_n^{(2)*}$

$$\xi_n^{(0)*} = \frac{2d}{d+4} \quad ; \quad \xi_T^{(0)*} = \frac{2}{d+4} \quad ; \quad V_q^* = \frac{m^2\beta^2}{d(d+2)V_0 n}\int_{\mathbb{R}^d} dv\, V^4 J[S_2(c^2)\mathcal{M}(V)]$$
$$\xi_{T,1}^{(2)*} = \frac{dnk_BT}{(d-1)K_0}\xi_{T,1}^{(2)} = C_T^{(2)*}\frac{d(d+2)}{(d-1)(d+4)} \quad ; \quad C_T^{(2)*} = C_T^{(2)}\frac{nk_BTV_0}{K_0}$$
$$\xi_{T,2}^{(2)*} = \frac{dn^2k_B}{(d-1)K_0}\xi_{T,2}^{(2)} = C_n^{(2)*}\frac{d(d+2)}{(d-1)(d+4)} \quad ; \quad C_n^{(2)*} = C_n^{(2)}\frac{n^2k_BV_0}{K_0}$$

①

~~Gaz granulaire:~~

~~$$\xi^{(2)} = (1-\alpha^2)\frac{2\beta}{3n}\left[\omega[f^{(1)},f^{(1)}] + 2\omega[f^{(0)},f^{(2)}]\right]$$~~

~~$$\xi(f) = (1-\alpha^2)\frac{2}{3nk_BT}\omega[f,f]$$~~

Taux de déclin PBA:

$$\xi_n^{(0)} = \frac{1}{n}\,\omega[f^{(0)},f^{(0)}]$$

$$\xi_{u_i}^{(0)} = \frac{1}{nV_T}\,\omega[f^{(0)},V_i f^{(0)}]$$

$$\xi_T^{(0)} = \frac{m}{nk_BTd}\,\omega[f^{(0)},V^2 f^{(0)}] - \xi_n^{(0)}$$

A l'ordre 2:

$$\xi_n^{(2)} = \frac{2}{n}\omega[f^{(0)},f^{(2)}] + \frac{1}{n}\omega[f^{(1)},f^{(1)}]$$

$$\xi_{u_i}^{(2)} = \frac{1}{nV_T}\omega[f^{(0)},V_i f^{(2)}] + \frac{1}{nV_T}\omega[f^{(1)},V_i f^{(1)}] + \frac{1}{nV_T}\omega[f^{(2)},V_i f^{(0)}]$$

$$\xi_T^{(2)} = \frac{m}{nk_BTd}\omega[f^{(0)},V^2 f^{(2)}] + \frac{m}{nk_BTd}\omega[f^{(2)},V^2 f^{(0)}] - \xi_n^{(2)}$$

Contributions à l'ordre 1: doivent être linéaires dans les gradients: ce sont les seules qui contribuent à l'analyse de stabilité linéaire, i.e.

$$f^{(1)}f^{(1)} = O(\delta^2) \quad \text{(quadratique dans les gradients)}$$

$$f^{(2)} = f_L^{(2)} + f_{nL}^{(2)}$$

: $f_L^{(2)}$ est lin. ds gradients
$f_{nL}^{(2)}$ est non. lin. "

$\Rightarrow$

$$\boxed{\begin{aligned}\xi_n^{(2)} &= \frac{2}{n}\omega[f^{(0)},f_L^{(2)}]\\ \xi_{u_i}^{(2)} &= \frac{1}{nV_T}\omega[f^{(0)},V_i f_L^{(2)}] + \frac{1}{nV_T}\omega[f^{(2)},V_i f^{(0)}]\\ \xi_T^{(2)} &= \frac{m}{nk_BTd}\omega[f^{(0)},V^2 f_L^{(2)}] + \frac{m}{nk_BTd}\omega[f_L^{(2)},V^2 f^{(0)}] - \xi_n^{(2)}\end{aligned}}$$

Eq. de Chap.-Enskog pour $f^{(2)}$: collecte les termes d'ordre 2 dans l'Eq. de Boltz. (46):

~~$[\partial_t^{(0)}+J]f^{(2)} = -\partial_t^{(2)}f^{(0)}$~~

$$\partial_t^{(0)}f^{(2)} + \partial_t^{(1)}f^{(1)} + \partial_t^{(2)}f^{(0)} + V_1\cdot\nabla_1 f^{(1)} = p\,J_a[f^{(0)},f^{(2)}] + p\,J_a[\overset{\nearrow \text{ordre 2}}{\cancel{f^{(1)},f^{(1)}}}] + p\,J_a[f^{(2)},f^{(0)}]$$
$$\underbrace{+(1-p)J_c[f^{(0)},f^{(2)}] + (1-p)J_c[\overset{\nearrow \text{ordre 2}}{\cancel{f^{(1)},f^{(1)}}}] + (1-p)J_c[f^{(2)},f^{(0)}]}_{:=-Jf^{(2)}}$$

$$\Rightarrow [\partial_t^{(0)}+J]f^{(2)} = -\partial_t^{(2)}f^{(0)} - V_1\cdot\nabla_1 f^{(1)} - \partial_t^{(1)}f^{(1)}$$

~~Pour la contribution $f_L^{(2)}$~~

$$= -\partial_t^{(2)}f^{(0)} - (\partial_t^{(1)} + V_1\cdot\nabla_1)f^{(1)}$$

On a:

$$-\partial_t^{(2)} f^{(0)} = -\underbrace{\frac{\partial f^{(0)}}{\partial n}}_{=\frac{1}{n}f^{(0)}} \frac{\partial n}{\partial_t^{(2)}} - \underbrace{\frac{\partial f^{(0)}}{\partial u_i}}_{=-\frac{\partial f^{(0)}}{\partial V_i}} \frac{\partial u_i}{\partial_t^{(2)}} - \frac{\partial f^{(0)}}{\partial T} \frac{\partial T}{\partial^{(2)} t}$$

$$= -\frac{1}{n} f^{(0)} \underbrace{\frac{\partial n}{\partial^{(2)} t}}_{\overset{(6a)}{=} -\underbrace{\nabla_i (n u_i)}_{=O(\Delta)} - p n \xi_n^{(2)}} + \frac{\partial f^{(0)}}{\partial V_i} \frac{\partial u_i}{\partial^{(2)} t} - \frac{\partial f^{(0)}}{\partial T} \frac{\partial T}{\partial^{(2)} t}$$

$$= + p f^{(0)} \xi_n^{(2)} + \frac{\partial f^{(0)}}{\partial V_i} \frac{\partial u_i}{\partial^{(2)} t} - \frac{\partial f^{(0)}}{\partial T} \frac{\partial T}{\partial^{(2)} t}$$

avec:

$$\frac{\partial u_i}{\partial^{(2)} t} \overset{(6b)}{=} -\frac{1}{mn} \nabla_j P_{ij} - \underbrace{u_j \nabla_j u_i}_{=O(\nabla)} - p V_T \xi_{ui}^{(2)} \qquad ; P_{ij} = p^{(0)} \delta_{ij} - \eta \left(\nabla_i u_j + \nabla_j u_i - \frac{2}{d} \delta_{ij} \nabla_k u_k\right)$$

$$\frac{\partial T}{\partial^{(2)} t} \overset{(6c)}{=} - u_j \nabla_j T - \frac{2}{n k_B d} \left(P_{ij} \nabla_i u_j + \nabla_j q_j\right) - p T \xi_T^{(2)} \quad ; \quad q_j = -\kappa \nabla_j T - \mu \nabla_j n$$

ainsi:

$$\frac{\partial u_i}{\partial^{(2)} t} = -\frac{1}{mn}\Big[\underbrace{\nabla_j p^{(0)} \delta_{ij}}_{=O(\Delta)} - \nabla_j \eta \left(\nabla_i u_j + \nabla_j u_i - \frac{2}{d} \delta_{ij} \nabla_k u_k\right)\Big] - p V_T \xi_{ui}^{(2)}$$

$$= - p V_T \xi_{ui}^{(2)} + \frac{1}{mn} \eta \left(\nabla_i \nabla_j u_j + \nabla_j \nabla_j u_i - \frac{2}{d} \nabla_i \nabla_k u_k \right)$$

$$= - p V_T \xi_{ui}^{(2)} + \frac{\eta}{mn} \left(\frac{d-2}{d} \nabla_i \nabla_j u_j + \nabla_j \nabla_j u_i \right) + O(\Delta^3)$$

$$\frac{\partial T}{\partial^{(2)} t} = -\underbrace{u_j \nabla_j T}_{=O(\Delta)} - p T \xi_T^{(2)} - \frac{2}{n k_B d} \Big[\overbrace{p^{(0)} \delta_{ij} \nabla_i u_j}^{=O(\Delta)} - \eta \left(\nabla_i u_j + \nabla_j u_i - \frac{2}{d} \delta_{ij} \nabla_k u_k\right) \nabla_i u_j$$
$$- \kappa \nabla_j \nabla_j T - \mu \nabla_j \nabla_j n \Big] + O(\nabla^3)$$

$$= - p T \xi_T^{(2)} - \frac{2}{n k_B} \Big[-\eta \left(\nabla_i \nabla_i u_j u_j + \frac{d}{d} \nabla_i \nabla_j u_i u_j - \frac{2}{d} \nabla_i \nabla_k u_i u_k \right)$$
$$- \kappa \nabla_j \nabla_j T - \mu \nabla_j \nabla_j n \Big]$$

$$= - p T \xi_T^{(2)} + \frac{2}{n k_B d} \left[\kappa \nabla_j \nabla_j T + \mu \nabla_j \nabla_j n + \eta \left(\frac{d-2}{d} \nabla_i \nabla_i u_i u_j + \nabla_i \nabla_i u_j u_j \right) \right]$$

Conclusion:

$$\boxed{\begin{aligned} -\partial_t^{(2)} f^{(0)} = {} & p f^{(0)} \xi_n^{(2)} + \frac{\partial f^{(0)}}{\partial V_i} \left[- p V_T \xi_{ui}^{(2)} + \frac{1}{mn} \eta \left(\frac{d-2}{d} \nabla_i \nabla_j u_j + \nabla_j \nabla_j u_i \right) \right] \\ & + p T \xi_T^{(2)} \frac{\partial f^{(0)}}{\partial T} - \frac{\partial f^{(0)}}{\partial T} \left[\kappa \nabla_j \nabla_j T + \mu \nabla_j \nabla_j n + \eta \left(\frac{d-2}{d} \nabla_i \nabla_j u_i u_j + \nabla_i \nabla_i u_j u_j \right) \right] \frac{2}{n k_B d} \end{aligned}}$$

Il reste le terme $(\partial_t^{(1)} + V_1 \nabla) f^{(1)}$: comme la forme de $f^{(1)}$ est chez moi exactement la même que chez Santos, on doit avoir formellement le même résultat. ~~Conclusion~~:

~~$[\partial_t^{(0)} + J] f^{(2)} = -\partial_t^{(2)} f^{(0)} - (\partial_t^{(1)} + V_1 \cdot \nabla) f^{(1)}$~~

~~[illegible]~~:

Ainsi:

$$v_1\cdot\nabla f^{(1)} = -\frac{\beta^3}{n}\mathcal{M}(V)\Big[\frac{2m}{d+2}S_i(V)V_j\left(\kappa\nabla_i\nabla_jT+\mu\nabla_i\nabla_jn\right) + \frac{\eta}{\beta}u_k\nabla_k D_{ij}(V)\nabla_ju_i\Big]$$

$$= -\frac{\beta^3}{n}\mathcal{M}(V)\frac{2m}{d+2}S_i(V)V_j\left(\kappa\nabla_i\nabla_jT+\mu\nabla_i\nabla_jn\right) - \frac{\beta^2}{n}\eta D_{ij}(V)\mathcal{M}(V)\overset{V=v-u}{\boxed{V_k}}\nabla_k\nabla_ju_i$$

$\vdots$

etc.

On obtient:

$$\left[\partial_t^{(0)}+J\right]f^{(2)} = -\partial_t^{(2)}f^{(0)} - \left(\partial_t^{(1)}+v\cdot\nabla\right)f^{(1)}$$

$\Rightarrow$

$$\left[\partial_t^{(0)}+J\right]f^{(2)} \quad -p\,\xi_n^{(2)}f^{(0)}+p\,V_i\,\xi_{u_i}^{(2)}\frac{\partial f^{(0)}}{\partial V_i} \quad -p\,T\xi_T^{(2)}\frac{\partial f^{(0)}}{\partial T}$$

$$= \frac{1}{mn}\eta\left(\frac{d-2}{d}\nabla_i\nabla_ju_i+\nabla_j\nabla_ju_i\right)\frac{\partial f^{(0)}}{\partial V_i} - \frac{2}{nk_BTd}\left[\kappa\nabla_j\nabla_jT+\mu\nabla_j\nabla_jn+\overbrace{\eta\left(\frac{d-2}{d}\nabla_i\cancel{\nabla_j}\overset{u_i}{u_j}+\nabla_j\nabla_j\overset{u_j}{u_i}\right)}^{\text{non linéaire en } u \,\to\, \text{négligé}}\right]\frac{\partial f^{(0)}}{\partial T}$$

$$+\frac{2m}{d+2}\frac{\beta^3}{n}S_i(V)\,V_j\left(\kappa\nabla_i\nabla_jT+\mu\nabla_i\nabla_jn\right)\mathcal{M}(V)$$

$$-\frac{\beta^2}{n}\eta D_{ij}(V)\mathcal{M}(V)\cancel{V_k\nabla_k}\nabla_ju_i - \underset{?}{\underbrace{\frac{\beta^2}{n}\eta\cancel{D_{ij}(V)\frac{1}{mn}}\nabla_i\nabla_jp^{(0)}\mathcal{M}(V)}}$$

$$-\underset{?}{\underbrace{\frac{2m}{d+2}\frac{\beta^3}{n}\left(\overset{?}{\boxed{\frac{2T}{d}}}\kappa+n\mu\right)S_i(V)\cancel{\nabla_i}\nabla_j\,u_j\,\mathcal{M}(V)}}$$

Simplifications: p. 4651 Brey: $\xi^{(2)}$ sont des scalaires $\Rightarrow$ seules les contributions sous trace et scalaires contribuent à $f_L^{(2)}$. Notons $\delta f_L^{(2)}$ la partie résiduelle de $f_L^{(2)}$ menant à $\xi^{(2)}$, alors en négligeant les contributions vectorielles et en prenant la trace:

$$\left[\partial_t^{(0)}+J\right]\delta f_L^{(2)} - p\,\xi_n^{(2)}f^{(0)} + p\,V_i\,\xi_{u_i}^{(2)}\overset{?}{\boxed{\frac{\partial f^{(0)}}{\partial V_i}}} - p\,T\xi_T^{(2)}\frac{\partial f^{(0)}}{\partial T}$$

$$= -\frac{2}{nk_BTd}\left(\kappa\nabla_j\nabla_jT+\mu\nabla_j\nabla_jn\right)\frac{\partial f^{(0)}}{\partial T} + \frac{2m}{d+2}\frac{\beta^3}{n}\frac{1}{d}S_i(V)V_i\mathcal{M}(V)\left(\kappa\nabla_j\nabla_jT+\mu\nabla_j\nabla_jn\right)$$

$\Rightarrow$

$$\boxed{\begin{aligned}&\left[\partial_t^{(0)}+J\right]\delta f_L^{(2)} - p\,\xi_n^{(2)}f^{(0)} - p\,T\xi_T^{(2)}\frac{\partial f^{(0)}}{\partial T} + \overbrace{p\,V_i\,\xi_{u_i}^{(2)}\frac{\partial f^{(0)}}{\partial V_i}}^{=\,d\,\xi_u^{(2)}\frac{\partial f^{(0)}}{\partial V}}\\ &\quad= \Big(\kappa\underbrace{\nabla_i\nabla_i}_{=\nabla^2}T+\mu\underbrace{\nabla_i\nabla_i}_{=\nabla^2}n\Big)\left[\frac{2m}{d(d+2)}\frac{\beta^3}{n}S_j(V)V_j\mathcal{M}(V) - \frac{2}{nk_BTd}\frac{\partial f^{(0)}}{\partial T}\right]\end{aligned}} \qquad (1)$$

Solution $\delta f_L^{(2)}$ a la forme:

$$\delta f_L^{(2)} = M(\overset{n,}{T},V)\,\nabla^2T + N(\overset{n,}{T},V)\,\nabla^2 n$$

que l'on substitue dans les taux de déclin à l'ordre 2, ce qui mène à

$$\xi_n^{(2)} = \xi_{n,1}^{(2)}\,\nabla^2T + \xi_{n,2}^{(2)}\,\nabla^2n$$
$$\xi_u^{(2)} = \xi_{u,1}^{(2)}\,\nabla^2T + \xi_{u,2}^{(2)}\,\nabla^2n$$
$$\xi_T^{(2)} = \xi_{T,1}^{(2)}\,\nabla^2T + \xi_{T,2}^{(2)}\,\nabla^2n$$

Les $\xi_{A_i}^{(2)}$ seront simplement trouvés ensuite avec la connaissance de M et N. On doit évaluer:

$$\partial_t^{(0)}\delta f_L^{(2)} = \partial_t^{(0)}M(\overset{n,}{T},V)\,\nabla^2T + \partial_t^{(0)}N(\overset{n,}{T},V)\,\nabla^2n$$
$$= M(\overset{n}{T},V)\,\nabla^2\partial_t^{(0)}T + N(\overset{m}{T},V)\,\nabla^2\partial_t^{(0)}n$$

Analyse de dépendance fonctionelle :

$$\xi_n^{(0)} \sim n T^{1/2} \; ; \quad \partial_t^{(0)} n = -p\, n\, \xi_n^{(0)}$$

$$\xi_T^{(0)} \sim n T^{1/2} \; ; \quad \partial_t^{(0)} T = -p\, T\, \xi_T^{(0)}$$

Ainsi :

$$\nabla^2 \partial_t^{(0)} T = -p \nabla^2 \left[T \xi_T^{(0)} \right] \qquad ; \xi_T^{(0)} = \text{cte } n T^{1/2}$$

$$= -p \nabla^2 \left[n T^{3/2} \right] \text{cte}$$

$$= -p \text{ cte } \nabla \left[T^{3/2} \nabla n + \frac{3}{2} n T^{1/2} \nabla T \right]$$

$$= -p \text{ cte} \left[T^{3/2} \nabla^2 n + \frac{3}{2} n T^{1/2} \nabla^2 T + \frac{3}{2} T^{1/2} \nabla T \nabla n + \frac{3}{2} T^{1/2} \nabla n \nabla T + \frac{3}{2}\frac{1}{2} n T^{-1/2} (\nabla T)^2 \right]$$

$$= -p \left[\underbrace{\text{cte } n T^{1/2}}_{=\xi_T^{(0)}} \frac{T^{2/2}}{n} \nabla^2 n + \underbrace{\text{cte } n T^{1/2}}_{=\xi_T^{(0)}} \frac{3}{2} \nabla^2 T + \frac{3}{2} \underbrace{\text{cte } n T^{1/2}}_{=\xi_T^{(0)}} \frac{1}{n} \nabla T \nabla n + \frac{3}{2} \underbrace{\text{cte } n T^{1/2}}_{=\xi_T^{(0)}} \frac{1}{n} \nabla n \nabla T + \frac{3}{4} \underbrace{n T^{1/2} \text{cte}}_{=\xi_T^{(0)}} \frac{1}{T} (\nabla T)^2 \right]$$

$$= -p\, \xi_T^{(0)} \left[\frac{T}{n} \nabla^2 n + \frac{3}{2} \nabla^2 T + \underbrace{\frac{3}{2n} \nabla n \nabla T + \frac{3}{2n} \nabla n \nabla T + \frac{3}{4} \frac{1}{T} (\nabla T)^2}_{\text{termes non linéaires}} \right]$$

$$= -p\, \xi_T^{(0)} \left(\frac{T}{n} \nabla^2 n + \frac{3}{2} \nabla^2 T \right)$$

$$\nabla^2 \partial_t^{(0)} n = -p \nabla^2 \left[n \xi_n^{(0)} \right]$$

$$= -p \text{ cte } \nabla^2 \left[n^2 T^{1/2} \right]$$

$$= -p \text{ cte } \nabla \left[2n \nabla n T^{1/2} + n^2 \frac{1}{2} \frac{1}{T^{1/2}} \nabla T \right]$$

$$= -p \text{ cte} \left[2 \nabla n \nabla n T^{1/2} + 2n \nabla^2 n T^{1/2} + \cancel{2} n \nabla n \cancel{\tfrac{1}{2}} T^{-1/2} \nabla T + \cancel{2} n \nabla n \cancel{\tfrac{1}{2}} T^{-1/2} \nabla T \right.$$
$$\left. + n^2 \frac{1}{2} T^{-3/2} \left(-\frac{1}{2}\right) \nabla T \nabla T + n^2 \frac{1}{2} T^{-1/2} \nabla^2 T \right]$$

$$= -p \text{ cte} \left[2 T^{1/2} (\nabla n)^2 + 2n T^{1/2} \nabla^2 n + n T^{-1/2} \nabla n \nabla T + n T^{-1/2} \nabla n \nabla T \right.$$
$$\left. - \frac{1}{4} n^2 T^{-3/2} (\nabla T)^2 + \frac{1}{2} n^2 T^{-1/2} \nabla^2 T \right]$$

$$= -p \left[2 \underbrace{\text{cte } n T^{1/2}} \frac{1}{n} (\nabla n)^2 + 2 \underbrace{\text{cte } n T^{1/2}} \nabla^2 n + \underbrace{\text{cte } n T^{1/2}} \frac{1}{T} \nabla n \nabla T + \underbrace{\text{cte } n T^{1/2}} \frac{1}{T} \nabla n \nabla T \right.$$
$$\left. - \frac{1}{4} \underbrace{\text{cte } n T^{1/2}} \frac{n}{T^2} (\nabla T)^2 + \frac{1}{2} \underbrace{\text{cte } n T^{1/2}} \frac{n}{T} \nabla^2 T \right]$$

$$= -p\, \xi_n^{(0)} \left[\underbrace{\frac{2}{n} (\nabla n)^2}_{n.\text{lin.}} + 2 \nabla^2 n + \underbrace{\frac{1}{T} \nabla n \nabla T}_{n.\text{lin.}} + \underbrace{\frac{1}{T} \nabla n \nabla T}_{n.\text{lin.}} - \underbrace{\frac{n}{4T^2} (\nabla T)^2}_{n.\text{lin.}} + \frac{n}{2T} \nabla^2 T \right]$$

$$= -p\, \xi_n^{(0)} \left[2 \nabla^2 n + \frac{n}{2T} \nabla^2 T \right]$$

~~Ainsi :~~

~~$\partial_t^{(0)} \delta f_L^{(2)} = M(T,V) \left(-p \xi_T^{(0)}\right) \left(\frac{T}{n} \nabla^2 n + \frac{3}{2} \nabla^2 T \right) + N(T,V) \left(-p \xi_n^{(0)}\right) \left(2 \nabla^2 n + \frac{n}{2T} \nabla^2 T \right)$~~

~~Les eqs. par M et N deviennent donc :~~

~~$-p \xi_T^{(0)} M(T,V) \frac{T}{n} \nabla^2 n - p \xi_T^{(0)} M(T,V) \frac{3}{2} \nabla^2 T - p \xi_n^{(0)} N(T,V)\, 2 \nabla^2 n - p \xi_n^{(0)} M(T,V) \frac{n}{2T} \nabla^2 T$~~

~~$+ \left[J M(T,V) \right] \nabla^2 T + \left[J N(T,V) \right] \nabla^2 n = \left(K \nabla^2 T + n \nabla^2 n \right) \left[\frac{2m}{d(d+2)} \frac{\beta^3}{n} S_j(V) V_j M(V) - \frac{2}{n k_B d} \frac{\partial f^{(0)}}{\partial T} \right]$~~

~~Collecte les termes devant $\nabla^2 n$ et $\nabla^2 T$:~~

$$-p\, \xi_n^{(2)} f^{(0)} - p\, T\, \xi_T^{(2)} \frac{\partial f^{(0)}}{\partial T} + p V_T\, \xi_{u_i}^{(2)} \frac{\partial f^{(0)}}{\partial V_i}$$

De plus:

$$\partial_t^{(0)} M(n,T,V) = \frac{\partial M}{\partial n}\partial_t^{(0)} n + \frac{\partial M}{\partial T}\partial_t^{(0)} T$$

$$= \left[\frac{\partial}{\partial n}\left(-\rho\, n\, \xi_n^{(0)}\right) + \frac{\partial}{\partial T}\left(-\rho\, T\, \xi_T^{(0)}\right)\right] M$$

$$\partial_t^{(0)} N(n,T,V) = \left[\frac{\partial}{\partial n}\left(-\rho\, n\, \xi_n^{(0)}\right) + \frac{\partial}{\partial T}\left(-\rho\, T\, \xi_T^{(0)}\right)\right] N$$

Ainsi:

$$\partial_t^{(0)} \delta f_L^{(2)} = -\rho\, \xi_T^{(0)} M(n,T,V)\left(\frac{T}{n}\nabla^2 n + \frac{3}{2}\nabla^2 T\right) - \rho\, \xi_n^{(0)} N(n,T,V)\left(2\nabla^2 n + \frac{n}{2T}\nabla^2 T\right)$$
$$- \rho\left\{\left[n\,\xi_n^{(0)}\frac{\partial}{\partial n} + T\,\xi_T^{(0)}\frac{\partial}{\partial T}\right] M(n,T,V)\right\}\nabla^2 T - \rho\left\{\left[n\,\xi_n^{(0)}\frac{\partial}{\partial n} + T\,\xi_T^{(0)}\frac{\partial}{\partial T}\right] N(n,T,V)\right\}\nabla^2 n$$

En remplaçat dans l'Eq. (1) on a donc

$$-\rho\,\xi_T^{(0)} M\,\frac{T}{n}\nabla^2 n - \rho\,\xi_T^{(0)} M\,\frac{3}{2}\nabla^2 T - \rho\,\xi_n^{(0)} N\; 2\nabla^2 n - \rho\,\xi_n^{(0)} N\,\frac{n}{2T}\nabla^2 T$$
$$-\rho\,\nabla^2 T\left(n\,\xi_n^{(0)}\partial_n + T\,\xi_T^{(0)}\partial_T\right) M - \rho\,\nabla^2 n\left(n\,\xi_n^{(0)}\partial_n + T\,\xi_T^{(0)}\partial_T\right) N$$
$$+\nabla^2 T\,(JM) + \nabla^2 n\,(JN) \quad -\rho\,\xi_n^{(2)} f^{(0)} - \rho\, T\,\xi_T^{(2)}\frac{\partial f^{(0)}}{\partial T} + \rho\, V_T\,\xi_{ui}^{(2)}\frac{\partial f^{(0)}}{\partial V_i}$$
$$= \left(K\nabla^2 T + \mu\nabla^2 n\right)\left[\frac{2m}{d(d+2)}\frac{\beta^3}{n} S_j(V)\, V_j\, \mathcal{M}(V) - \frac{2}{n k_B d}\frac{\partial f^{(0)}}{\partial T}\right]$$

On collecte les termes en $\nabla^2 n$ et $\nabla^2 T$:

$\nabla^2 n$:

$$-\rho\,\xi_T^{(0)} M\frac{T}{n} - \rho\,\xi_n^{(0)} N\, 2 - \rho\left(n\,\xi_n^{(0)}\partial_n + T\,\xi_T^{(0)}\partial_T\right) N + (JN)$$
$$-\rho\,\xi_{n,2}^{(2)} f^{(0)} - \rho\, T\,\xi_{T,2}^{(2)}\partial_T f^{(0)} + \rho\, V_T\,\xi_{ui,2}^{(2)}\partial_{V_i} f^{(0)} = \mu\left[\frac{2m}{d(d+2)}\frac{\beta^3}{n} S_j(V)\, V_j\, \mathcal{M}(V) - \frac{2}{n k_B d}\partial_T f^{(0)}\right]$$

$\nabla^2 T$:

$$-\rho\,\xi_T^{(0)} M\frac{3}{2} - \rho\,\xi_n^{(0)} N\frac{n}{2T} - \rho\left(n\,\xi_n^{(0)}\partial_n + T\,\xi_T^{(0)}\partial_T\right) M + (JM)$$
$$-\rho\,\xi_{n,1}^{(2)} f^{(0)} - \rho\, T\,\xi_{T,1}^{(2)}\partial_T f^{(0)} + \rho\, V_T\,\xi_{ui,1}^{(2)}\partial_{V_i} f^{(0)} = K\left[\frac{2m}{d(d+2)}\frac{\beta^3}{n} S_j(V)\, V_j\, \mathcal{M}(V) - \frac{2}{n k_B d}\partial_T f^{(0)}\right]$$

$\Rightarrow$

$$\boxed{\begin{aligned}
&\left(-\rho\, n\,\xi_n^{(0)}\partial_n - \rho\, T\,\xi_T^{(0)}\partial_T - \rho\,\frac{3}{2}\xi_T^{(0)} + J\right) M - \rho\,\xi_n^{(0)}\frac{n}{2T} N - \rho\,\xi_{n,1}^{(2)} f^{(0)} - \rho\, T\,\xi_{T,1}^{(2)}\partial_T f^{(0)} + \rho\, V_T\,\xi_{ui,1}^{(2)}\partial_{V_i} f^{(0)}\\
&\qquad = K\left[\frac{2m}{d(d+2)}\frac{\beta^3}{n} S_j(V)\, V_j\, \mathcal{M}(V) - \frac{2}{n k_B d}\partial_T f^{(0)}\right]\\
&\left(-\rho\, n\,\xi_n^{(0)}\partial_n - \rho\, T\,\xi_T^{(0)}\partial_T - \rho\, 2\,\xi_n^{(0)} + J\right) N - \rho\,\xi_T^{(0)}\frac{T}{n} M - \rho\,\xi_{n,2}^{(2)} f^{(0)} - \rho\, T\,\xi_{T,2}^{(2)}\partial_T f^{(0)} + \rho\, V_T\,\xi_{ui,2}^{(2)}\partial_{V_i} f^{(0)}\\
&\qquad = \mu\left[\frac{2m}{d(d+2)}\frac{\beta^3}{n} S_j(V)\, V_j\, \mathcal{M}(V) - \frac{2}{n k_B d}\partial_T f^{(0)}\right]
\end{aligned}}$$

On a bien la m. chose que (D9) et (D10) Brey dans les limites adéquates. (R)

Développement en polynômes de Sonine:

$$M(V) = C_T^{(2)}\, S_2(V^2/V_T^2)\,\mathcal{M}(V) \quad , \quad S_2(V^2/V_T^2) = S_2(c^2) = \frac{1}{2}c^4 - \frac{d+2}{2}c^2 + \frac{d(d+2)}{8}$$
$$N(V) = C_N^{(2)}\, S_2(V^2/V_T^2)\,\mathcal{M}(V)$$

$$V_T = \sqrt{2/\beta m}$$

$$\mathcal{M}(V) = 1/\pi^{d/2}\, e^{-c^2}\,\frac{n}{V_T^d}$$

$$S_i(V) = \left(\frac{m}{2}V^2 - \frac{d+2}{2}k_B T\right)V_i$$

Ceci permet d'évaluer les taux de déclin:

$$\begin{aligned}
\xi_n^{(2)} &= \frac{2}{n}\,\omega[f^{(0)}, f_2] \\
&= \frac{2}{n}\,\omega[f^{(0)}, \delta f_2^{(2)}] \\
&= \frac{2}{n}\,\omega\left[f^{(0)}, M(n,T,v)\nabla^2 T + N(n,T,v)\nabla^2 n\right] \\
&= \frac{2}{n}\,\omega[f^{(0)}, M]\nabla^2 T + \frac{2}{n}\,\omega[f^{(0)}, N]\nabla^2 n \\
&= \underbrace{\frac{2}{n}\,\omega[f^{(0)}, S_2\mathcal{M}]\,C_T^{(2)}}_{=\xi_{n,1}^{(2)}}\nabla^2 T + \underbrace{\frac{2}{n}\,\omega[f^{(0)}, S_2\mathcal{M}]\,C_n^{(2)}}_{=\xi_{n,2}^{(2)}}\nabla^2 n
\end{aligned}$$

Ainsi:

$$\boxed{\begin{aligned}
\xi_{n,1}^{(2)} &= \frac{2}{n}\,\omega[f^{(0)}, S_2\mathcal{M}]\,C_T^{(2)} && \propto C_T^{(2)} \\
\xi_{n,2}^{(2)} &= \frac{2}{n}\,\omega[f^{(0)}, S_2\mathcal{M}]\,C_n^{(2)} && \propto C_n^{(2)} \\
\xi_{a_i,1}^{(2)} &= \frac{1}{n v_T}\,\omega[f^{(0)}, V_i S_2\mathcal{M}]\,C_T^{(2)} + \frac{1}{n v_T}\,\omega[S_2\mathcal{M}, V_i f^{(0)}]\,C_T^{(2)} && \propto C_T^{(2)} \\
\xi_{a_i,2}^{(2)} &= \frac{1}{n v_T}\,\omega[f^{(0)}, V_i S_2\mathcal{M}]\,C_n^{(2)} + \frac{1}{n v_T}\,\omega[S_2\mathcal{M}, V_i f^{(0)}]\,C_n^{(2)} && \propto C_n^{(2)} \\
\xi_{T,1}^{(2)} &= \frac{m}{n k_B T d}\,\omega[f^{(0)}, V^2 S_2\mathcal{M}]\,C_T^{(2)} + \frac{m}{n k_B T d}\,\omega[S_2\mathcal{M}, V^2 f^{(0)}]\,C_T^{(2)} - \xi_{n,1}^{(2)} && \propto C_T^{(2)} \\
\xi_{T,2}^{(2)} &= \frac{m}{n k_B T d}\,\omega[f^{(0)}, V^2 S_2\mathcal{M}]\,C_n^{(2)} + \frac{m}{n k_B T d}\,\omega[S_2\mathcal{M}, V^2 f^{(0)}]\,C_n^{(2)} - \xi_{n,2}^{(2)} && \propto C_n^{(2)}
\end{aligned}} \quad (3)$$

A calculer

Insère forme de M et N dans (2), multiplie par V^4, et intègre sur V, avec $C_T^{(2)} \sim n^{-2} T^{-1}$ et $C_n^{(2)} \sim n^{-3} T^{0}$: (cf. plus loin pour cette dépendance fonctionnelle)

$$\int_{\mathbb{R}^d} dv\, V^4 \left[-p\, n\, \xi_n^{(0)}\partial_n - pT\xi_T^{(0)}\partial_T - p\frac{3}{2}\xi_T^{(0)} + J\right] C_T^{(2)} S_2(c^2)\mathcal{M}(V)$$

$$- p\,\xi_n^{(0)}\frac{n}{2T}\int_{\mathbb{R}^d} dv\, V^4\, C_n^{(2)} S_2(c^2)\mathcal{M}(V) - p\,\xi_{n,1}^{(2)}\int_{\mathbb{R}^d} dv\, V^4 f^{(0)} - p\,\xi_{T,1}^{(2)}\int_{\mathbb{R}^d} dv\, V^4 \underbrace{T\partial_T f^{(0)}}_{=-\frac{1}{2}\frac{\partial}{\partial V_i}(V_i f^{(0)})} + p\, v_T\, \xi_{a_i,1}^{(2)}\int_{\mathbb{R}^d} dv\, V^4 \partial_{V_i} f^{(0)}$$

$$= K\frac{2m}{d(d+2)}\frac{\beta^3}{n}\int_{\mathbb{R}^d} dv\, V^4 S_j(V) V_j \mathcal{M}(V) - K\frac{2}{n k_B d T}\int_{\mathbb{R}^d} dv\, V^4 \underbrace{T\partial_T f^{(0)}}_{=-\frac{1}{2}\frac{\partial}{\partial V_i}(V_i f^{(0)})}$$

avec:

$$\begin{aligned}
\partial_n C_T^{(2)} &= \partial_n\, \text{cte}\; n^{-2} T^{-1} = -\frac{2}{n} C_T^{(2)} \\
\partial_T C_T^{(2)} &= \partial_T\, \text{cte}\; n^{-3} T^{-1} = -\frac{1}{T^2}\,\text{cte}\; n^{-2} = -\frac{1}{T}\,\text{cte}\; n^{-6} T^{-1} = -\frac{1}{T} C_T^{(2)}
\end{aligned}$$

$$a_2 + 1 = \frac{4}{d(d+2)}\left(\frac{\beta m}{2}\right)^2 \frac{1}{n}\int_{\mathbb{R}^d} dv\, V^4 f^{(0)}$$

$$\Longrightarrow \int_{\mathbb{R}^d} dv\, V^4 f^{(0)} = \frac{d(d+2)}{4}\, n \left(\frac{2}{\beta m}\right)^2 (a_2 + 1)$$

il vient:

$$\int_{\mathbb{R}^d} dv\, v^4 \Big[\underbrace{-pT\xi_T^{(0)}\left(-\frac{1}{T}\right) - p\frac{3}{2}\xi_T^{(0)}}_{=-p\xi_T^{(0)}\frac{5}{2}} + J \Big] C_T^{(2)} S_2(c^2)\mathcal{M}(v) \qquad \Big] = -p\xi_T^{(0)}\frac{5}{2}C_T^{(2)}\int_{\mathbb{R}^d} dv\, v^4 S_2(c^2)\mathcal{M}(v) + C_T^{(2)}\int_{\mathbb{R}^d} dv\, v^4 J S_2(c^2)\mathcal{M}(v)$$

$$-p\xi_n^{(0)}\frac{n}{2T}C_n^{(2)}\int_{\mathbb{R}^d} dv\, v^4 S_2(c^2)\mathcal{M}(v) - p\xi_{n,1}^{(2)}\frac{d(d+2)}{4}n\left(\frac{2}{\beta m}\right)^2(a_2+1)$$

$$+p\frac{1}{2}\xi_{T,1}^{(2)}\int_{\mathbb{R}^d} dv\, v^4\frac{\partial}{\partial v_i}\left(v_i f^{(0)}\right) + p v_T \xi_{u_i}^{(1)}\int_{\mathbb{R}^d} dv\, v^4\frac{\partial}{\partial v_i}f^{(0)}$$

$$= K\frac{2m}{d(d+2)}\frac{\beta^3}{n}\int_{\mathbb{R}^d} dv\, v^4 S_j(v)v_j\mathcal{M}(v) + K\frac{1}{nk_BdT}\int_{\mathbb{R}^d} dv\, v^4\frac{\partial}{\partial v_i}\left(v_i f^{(0)}\right)$$

$$\Longrightarrow \Bigg[-p\overset{=\xi_T^{(0)*}}{\xi_T^{(0)}}\frac{5}{2}\overset{=1}{\frac{\int_{\mathbb{R}^d} dv\, v^4 S_2(c^2)\mathcal{M}(v)}{V_0\int_{\mathbb{R}^d} dv\, v^4 S_2(c^2)\mathcal{M}(v)}} + \overset{:=V_\xi^*}{\frac{\int_{\mathbb{R}^d} dv\, v^4 J S_2(c^2)\mathcal{M}(v)}{V_0\int_{\mathbb{R}^d} dv\, \cancel{v^4} S_2(c^2)\mathcal{M}(v)}} \Bigg] C_T^{(2)}$$

$2p\xi_n^{(0)*}$

$$-p\overset{=\xi_n^{(0)*}}{\xi_n^{(0)}}\frac{n}{2T}C_n^{(2)}\overset{=1}{\frac{\int_{\mathbb{R}^d} dv\, v^4 S_2(c^2)\mathcal{M}(v)}{V_0\int_{\mathbb{R}^d} dv\, v^4 S_2(c^2)\mathcal{M}(v)}} - p\xi_{n,1}^{(2)}\frac{d(d+2)}{4}n\left(\frac{2}{\beta m}\right)^2\frac{a_2+1}{V_0\int_{\mathbb{R}^d} dv\, v^4 S_2(c^2)\mathcal{M}(v)}$$

$$+p\frac{1}{2}\xi_{T,1}^{(2)}\frac{\int_{\mathbb{R}^d} dv\, v^4\frac{\partial}{\partial v_i}\left(v_i f^{(0)}\right)}{V_0\int_{\mathbb{R}^d} dv\, v^4 S_2(c^2)\mathcal{M}(v)} + p v_T\xi_{u_i}^{(1)}\cancelto{0}{\frac{\int_{\mathbb{R}^d} dv\, \overset{\text{impar}}{v^4\frac{\partial}{\partial v_i}f^{(0)}}}{V_0\int_{\mathbb{R}^d} dv\, v^4 S_2(c^2)\mathcal{M}(v)}}$$

$$= K\frac{2m}{d(d+2)}\frac{\beta^3}{n}\frac{\int_{\mathbb{R}^d} dv\, v^4 S_j(v)v_j\mathcal{M}(v)}{V_0\int_{\mathbb{R}^d} dv\, v^4 S_2(c^2)\mathcal{M}(v)} + K\frac{1}{nk_BTd}\frac{\int_{\mathbb{R}^d} dv\, v^4\frac{\partial}{\partial v_i}\left(v_i f^{(0)}\right)}{V_0\int_{\mathbb{R}^d} dv\, v^4 S_2(c^2)\mathcal{M}(v)}$$

$$\Longrightarrow \Big(\overset{2p\xi_n^{(0)*}}{-p\frac{5}{2}\xi_T^{(0)*}} + V_\xi^*\Big)C_T^{(2)} - p\xi_n^{(0)*}\frac{n}{2T}C_n^{(2)} - p\xi_{n,1}^{(2)}\frac{d(d+2)}{4}n\left(\frac{2}{\beta m}\right)\frac{a_2+1}{V_0\int_{\mathbb{R}^d} dv\, v^4 S_2(c^2)\mathcal{M}(v)}$$

$$+p\frac{1}{2}\xi_{T,1}^{(2)}\frac{\boxed{\int_{\mathbb{R}^d} dv\, v^4\frac{\partial}{\partial v_i}\left(v_i f^{(0)}\right)}}{V_0\int_{\mathbb{R}^d} dv\, v^4 S_2(c^2)\mathcal{M}(v)} = K\frac{2m}{d(d+2)}\frac{\beta^3}{n}\frac{\int_{\mathbb{R}^d} dv\, v^4 S_j(v)v_j\mathcal{M}(v)}{V_0\int_{\mathbb{R}^d} dv\, v^4 S_2(c^2)\mathcal{M}(v)} + K\frac{1}{nk_BTd}\frac{\boxed{\int_{\mathbb{R}^d} dv\, v^4\frac{\partial}{\partial v_i}\left(v_i f^{(0)}\right)}}{V_0\int_{\mathbb{R}^d} dv\, v^4 S_2(c^2)\mathcal{M}(v)}$$

Avec :

$$V_0\int_{\mathbb{R}^d} dv\, v^4 S_2(c^2)\mathcal{M}(v) = V_0\frac{n}{v_T^d}\frac{1}{\pi^{d/2}}\int_{\mathbb{R}^d} dv\, v^4\left[\frac{1}{2}\frac{v^4}{v_T^4} - \frac{d+2}{2}\frac{v^2}{v_T^2} + \frac{d(d+2)}{8}\right]e^{-v^2/v_T^2} \quad ; \; \begin{array}{l} c = v/v_T \\ dv = v_T^d\, dc \end{array}$$

$$= V_0\frac{n}{\cancel{v_T^d}}\frac{1}{\pi^{d/2}}\cancel{v_T^d}\, v_T^4\int_{\mathbb{R}^d} dc\, e^{-c^2}\left(\frac{1}{2}c^8 - \frac{d+2}{2}c^6 + \frac{d(d+2)}{8}c^4\right) \quad ; \; v_T = \sqrt{2/\beta m}$$

$$= V_0 n\frac{1}{\pi^{d/2}}\frac{4}{\beta^2 m^2}\pi^{d/2}\left[\frac{1}{2}\frac{\Gamma\left(\frac{d+8}{2}\right)}{\Gamma(d/2)} - \frac{d+2}{2}\frac{\Gamma\left(\frac{d+6}{2}\right)}{\Gamma(d/2)} + \frac{d(d+2)}{8}\frac{\Gamma\left(\frac{d+4}{2}\right)}{\Gamma(d/2)}\right]$$

$$= V_0 n\frac{4k_B^2T^2}{m^2}\left[\frac{1}{2}\frac{d+6}{2}\frac{d+4}{2}\frac{d+2}{2}\frac{d}{2} - \frac{d+2}{2}\frac{d+4}{2}\frac{d+2}{2}\frac{d}{2} + \frac{d(d+2)}{8}\frac{d+2}{2}\frac{d}{2}\right]$$

$$= V_0 n\frac{4k_B^2T^2}{m^2}\frac{d(d+2)}{32}\left[(d+6)(d+4) - 2(d+4)(d+2) + d(d+2)\right]$$

$$= V_0 n\frac{k_B^2T^2}{m^2}\frac{d(d+2)}{8}\left[\cancel{d^2} + \cancel{4d} + \cancel{6d} + 24 - \cancel{2d^2} - \cancel{4d} - \cancel{8d} - 16 + \cancel{d^2} + \cancel{2d}\right]$$

$$= \frac{d(d+2)V_0 n k_B^2T^2}{m^2}$$

: <u>OK</u> (Brey: de la déf. de S_2, il a un facteur ×2)

$$\int_{\mathbb{R}^d} dv\, v^4 \underbrace{\frac{\partial}{\partial \bar{v}_i}\left(\bar{v}_i f^{(0)}\right)}_{= d f^{(0)} + v_i \frac{\partial f^{(0)}}{\partial \bar{v}_i}} = d \int_{\mathbb{R}^d} dv\, v^4 f^{(0)} + \int_{\mathbb{R}^d} dv\, v^4 v_i \frac{\partial f^{(0)}}{\partial \bar{v}_i}$$

$v^2 = \sum_i v_i^2 \;;\; v^4 = (v^2)^2 = (\sum_i v_i^2)^2 = \sum_{ij} v_i^2 v_j^2$

$$= \frac{d^2(d+2)}{4}\, n \left(\frac{2}{\beta m}\right)^2 (a_2+1) + \left[- \int_{\mathbb{R}^d} dv \frac{\partial}{\partial v_i}\left(v^4 v_i\right) f^{(0)} + \underbrace{v^4 v_i f^{(0)} \Big|_{-\infty}^{\infty}}_{=0} \right]$$

$$= \frac{d^2(d+2)}{4}\, n \left(\frac{2}{\beta m}\right)^2 (a_2+1) - \int_{\mathbb{R}^d} dv \left[v^4 \frac{\partial v_i}{\partial v_i} + v_i \frac{\partial}{\partial v_i} \sum_{jk} v_j^2 v_k^2 \right] f^{(0)}$$

$$= \frac{d^2(d+2)}{\cancel{4}}\, n\, \frac{\cancel{4} k_B^2 T^2}{m^2} (a_2+1) - \int_{\mathbb{R}^d} dv \left[d v^4 + \underbrace{v_i \frac{\partial}{\partial v_i} v_j v_j v_k v_k}_{\substack{= 2 v_j \delta_{ij} v_k v_k + 2 v_j v_j v_k \delta_{ik} \\ = 2 v_i v^2 + 2 v_i v^2 \\ = 4 v_i v^2}} \right] f^{(0)}$$

$$= d^2(d+2)\, \frac{n k_B^2 T^2}{m^2} (a_2+1) - \int_{\mathbb{R}^d} dv \left[d v^4 + 4 v^4 \right] f^{(0)}$$

$$= d^2(d+2)\, \frac{n k_B^2 T^2}{m^2} (a_2+1) - (d+4) \underbrace{\int_{\mathbb{R}^d} dv\, v^4 f^{(0)}}_{= \frac{d(d+2)}{\cancel{4}}\, n\, \frac{\cancel{4} k_B^2 T^2}{m} (a_2+1)}$$

$$= d(d+2)\, \frac{n k_B^2 T^2}{m^2} (a_2+1) \left[d - (d+4) \right]$$

$$= -4 d(d+2)\, \frac{n k_B^2 T^2}{m^2} (a_2+1) \qquad \checkmark$$

$$\int_{\mathbb{R}^d} dv\, v^4 S_j(v)\, v_j\, \mathcal{M}(v) = \int_{\mathbb{R}^d} dv\, v^4 \left(\frac{m}{2} v^2 - \frac{d+2}{2} k_B T \right) v_j v_j \frac{n}{v_T^d} \frac{1}{\pi^{d/2}} e^{-v^2/v_T^2}$$

$$= \int_{\mathbb{R}^d} dv\, v^6 \frac{n}{v_T^d} \frac{1}{\pi^{d/2}} \frac{m}{2} \left(v^2 - \frac{d+2}{2} \overbrace{\frac{2 k_B T}{m}}^{= v_T^2} \right) e^{-v^2/v_T^2}$$

$$= \frac{n}{\cancel{v_T^d}} \frac{1}{\pi^{d/2}} \frac{m}{2} \cancel{v_T^d}\, v_T^6 \int_{\mathbb{R}^d} dc\, c^6\, v_T^2 \left(c^2 - \frac{d+2}{2} \right) e^{-c^2}$$

$$= \frac{n}{\pi^{d/2}} \frac{m}{2} v_T^8 \int_{\mathbb{R}^d} dc\, e^{-c^2} \left(c^8 - \frac{d+2}{2} c^6 \right)$$

$$= \frac{n}{\cancel{\pi^{d/2}}} \frac{m}{2} v_T^8\, \cancel{\pi^{d/2}} \left[\frac{\Gamma(\frac{d+8}{2})}{\Gamma(d/2)} - \frac{d+2}{2} \frac{\Gamma(\frac{d+6}{2})}{\Gamma(d/2)} \right]$$

$$= n \frac{m}{2} v_T^8 \left[\frac{d+6}{2} \frac{d+4}{2} \frac{d+2}{2} \frac{d}{2} - \frac{d+2}{2} \frac{d+4}{2} \frac{d+2}{2} \frac{d}{2} \right]$$

$$= n \frac{m}{2} v_T^8 \frac{1}{16} (d+4)(d+2) d \left[\overbrace{\cancel{d} + 6 - (\cancel{d} + 2)}^{=4} \right]$$

$$= n \frac{m}{8} v_T^8\, d(d+2)(d+4)$$

$$= n \frac{m}{8} \left(\frac{2}{\beta m}\right)^4 d(d+2)(d+4)$$

$$= \frac{2n}{\beta^4 m^3} d(d+2)(d+4) \qquad \checkmark$$

Mis tout ensemble:

$$\left(\overset{2p\xi_n^{(0)*}}{-p\frac{5}{2}\xi_T^{(0)*}}+\nu_\xi^*\right)C_T^{(2)}-p\xi_n^{(0)*}\frac{n}{2T}C_n^{(2)}-p\xi_{n,1}^{(2)}\frac{\cancel{d(d+2)}}{4}\frac{2\cancel{X}}{\cancel{\beta}\cancel{\sigma X}}(a_2+1)\frac{m^{\cancel{2}}}{\cancel{d(d+2)}\nu_0\cancel{n}k_B^{\cancel{2}}T^{\cancel{2}}}$$

$$+p\frac{1}{2}\xi_{T,1}^{(2)}\ (-4)\cancel{d(d+2)}\frac{\cancel{nk_B^2T^2}}{\cancel{m^2}}(a_2+1)\frac{\cancel{m^2}}{\cancel{d(d+2)}\nu_0\cancel{nk_B^2T^2}}$$

$$=K\frac{2\cancel{\sigma}}{d(d+2)}\frac{\cancel{\beta^2}}{\cancel{\beta}}\frac{2\cancel{X}}{\cancel{\beta X}\cancel{m^2}}\cancel{d(d+2)}(d+4)\frac{\cancel{m^2}}{\cancel{d(d+2)}\nu_0 nk_B^{\cancel{2}}T^{\cancel{2}}}+K\frac{1}{nk_BTd}(-4)\cancel{d(d+2)}\frac{\cancel{nk_B^2T^2}}{\cancel{m^2}}(a_2+1)\frac{\cancel{m^2}}{\cancel{d(d+2)}\nu_0\cancel{nk_B^2T^2}}$$

$\Rightarrow$

$$\left(\overset{2p\xi_n^{(0)*}}{-p\frac{5}{2}\xi_T^{(0)*}}+\nu_\xi^*\right)C_T^{(2)}-p\xi_n^{(0)*}\frac{n}{2T}C_n^{(2)}-p\xi_{n,1}^{(2)}\frac{a_2+1}{2}\frac{m\beta}{\nu_0}-p\xi_{T,1}^{(2)}2(a_2+1)\frac{1}{\nu_0}$$

$$=K\frac{4(d+4)}{d(d+2)}\frac{\beta}{\nu_0 n}-4K(a_2+1)\frac{\beta}{nd}$$

$\Rightarrow$

$$\left(-p\frac{5}{2}\xi_T^{(0)*}+\nu_\xi^*\right)C_T^{(2)}=p\xi_n^{(0)*}\frac{n}{2T}C_n^{(2)}+p\xi_{n,1}^{(2)}\frac{a_2+1}{2}\frac{m\beta}{\nu_0}+p\frac{\xi_{T,1}^{(2)}}{\nu_0}2(a_2+1)$$

$$+\frac{4K}{d(d+2)}\frac{\beta}{n\nu_0}\left[\underbrace{d+4-(d+2)(a_2+1)}_{\substack{=\cancel{d}+4-\cancel{d}-2-a_2(d+2)\\ =2-a_2(d+2)}}\right]$$

$$\Rightarrow\boxed{\left(\overset{2p\xi_n^{(0)*}}{-p\frac{5}{2}\xi_T^{(0)*}}+\nu_\xi^*\right)C_T^{(2)}=p\xi_n^{(0)*}\frac{n}{2T}C_n^{(2)}+p\xi_{n,1}^{(2)}\frac{a_2+1}{2}\frac{m\beta}{\nu_0}+p\frac{\xi_{T,1}^{(2)}}{\nu_0}2(a_2+1)+\frac{8K\beta}{d(d+2)n\nu_0}\left[1-\frac{d+2}{2}a_2\right]}\qquad(D.15)$$

Idem. Eq. D.15, avec facteur 1/2 et $c^*=2a_2$; c'est correct.

Seconde équation: insère M et N, multiplie par V^4 et intègre sur V, avec $C_T^{(2)}\sim n^0T^{-1}$ et $C_n^{(2)}\sim n^{-1}T^0$:

$$\int_{\mathbb{R}^d}dv\,V^4\left[-pn\xi_n^{(0)}\partial_n-pT\xi_T^{(0)}\partial_T-p2\xi_n^{(0)}+J\right]C_n^{(2)}S_2(c^2)\mathcal{M}(V)$$

$\hat{m}$. termes que avant

$$-p\xi_T^{(0)}\frac{T}{n}\int_{\mathbb{R}^d}dv\,V^4C_T^{(2)}S_2(c^2)\mathcal{M}(V)-p\xi_{n,2}^{(2)}\int_{\mathbb{R}^d}dv\,V^4f^{(0)}-p\xi_{T,2}^{(2)}\int_{\mathbb{R}^d}dv\,V^4T\partial_Tf^{(0)}+pV_T\xi_{u_i,2}^{(2)}\int_{\mathbb{R}^d}dv\,V^4\cancel{\frac{\partial}{\partial V_i}f^{(0)}}\to 0$$

$$=\mathcal{M}\frac{2m}{d(d+2)}\frac{\beta^3}{n}\int_{\mathbb{R}^d}dv\,V^4S_j(V)V_j\mathcal{M}(V)-\mathcal{M}\frac{2}{nk_BdT}\int_{\mathbb{R}^d}dv\,V^4T\partial_Tf^{(0)}$$

$\hat{m}$. termes que avant

avec :

$$\partial_nC_n^{(2)}=\partial_n\,\mathrm{cte}\,n^{-3}T^0=-\frac{3}{n^4}\mathrm{cte}\,T^0=-\frac{3}{n}C_n^{(2)}$$
$$\partial_TC_n^{(2)}=\partial_T\,\mathrm{cte}\,n^{-3}T^0=0$$

$$\Rightarrow\int_{\mathbb{R}^d}dv\,V^4\left[-p\cancel{n}\xi_n^{(0)}\left(-\frac{3}{\cancel{n}}\right)-p2\xi_n^{(0)}+J\right]C_n^{(2)}S_2(c^2)\mathcal{M}(V)=\dots$$

$$\Rightarrow\int_{\mathbb{R}^d}dv\,V^4\left[3p\xi_n^{(0)}-2p\xi_n^{(0)}+J\right]C_n^{(2)}S_2(c^2)\mathcal{M}(V)=\dots$$

$$\Rightarrow+p\xi_n^{(0)}C_n^{(2)}\int_{\mathbb{R}^d}dv\,V^4S_2(c^2)\mathcal{M}(V)+\int_{\mathbb{R}^d}dv\,V^4JC_n^{(2)}S_2(c^2)\mathcal{M}(V)=\dots$$

$$\Rightarrow\left(+p\xi_n^{(0)*}+\nu_\xi^*\right)C_n^{(2)}-p\xi_T^{(0)*}\frac{T}{n}C_T^{(2)}-p\xi_{n,2}^{(2)}\frac{a_2+1}{2}\frac{m\beta}{\nu_0}-p\xi_{T,2}^{(2)}2(a_2+1)\frac{1}{\nu_0}$$

$$=\frac{8\mathcal{M}\beta}{d(d+2)n\nu_0}\left[1-\frac{d+2}{2}a_2\right]\qquad(D.16)$$

$$\Rightarrow\boxed{\left(+p\xi_n^{(0)*}+\nu_\xi^*\right)C_n^{(2)}=p\xi_T^{(0)*}\frac{T}{n}C_T^{(2)}+p\xi_{n,2}^{(2)}\frac{a_2+1}{2}\frac{m\beta}{\nu_0}+p\frac{\xi_{T,2}^{(2)}}{\nu_0}2(a_2+1)+\frac{8\mathcal{M}\beta}{d(d+2)n\nu_0}\left[1-\frac{d+2}{2}a_2\right]}$$

Idem Brey, ~~sauf que lui a un facteur 2 à cause de la non-dépendance en n de M et N~~, i.e. $C_n^{(2)}$ ne dépend pas de n chez lui. ~~Je pense que je dois garder cette dépendance car n ≠ cte.~~ Erreur chez Brey.

Résumé :

$$V_\xi^* = \frac{\int_{\mathbb{R}^d} dv\, V^4\, J[S_2(c^2)\mathcal{M}(V)]}{V_0 \int_{\mathbb{R}^d} dv\, V^4 S_2(c^2)\mathcal{M}(V)} = \frac{m^2\beta^2}{d(d+2)V_0 n} \int_{\mathbb{R}^d} dv\, V^4\, J[S_2(c^2)\mathcal{M}(V)]$$

$\xi_{A,i}^{(2)}$ = Eq. (3) p.⑥

Eqs. : (D.15) et (D.16) : p.⑨ : 2 équations par les 2 inconnues $C_n^{(2)}$ et $C_T^{(2)}$ (Utiliser $\xi_{A,i}^{(2)}$)

Adimentionalisation des coefficients :

$$\boxed{\xi_{T,1}^{(2)*} = \frac{d\ nk_BT}{(d-1)\kappa_0}\,\xi_{T,1}^{(2)}}\ ,\quad \kappa_0 = \frac{d(d+2)}{2(d-1)}\frac{k_B}{m}\eta_0\ ;\ V_0 = \frac{nk_BT}{\eta_0}\ ;\ \eta_0 = \frac{d+2}{8}\frac{\Gamma(d/2)}{\pi^{(d-1)/2}}\frac{\sqrt{mkT}}{\sigma^{d-1}}$$

$$\boxed{\xi_{T,2}^{(2)*} = \frac{d\, n^2 k_B}{(d-1)\kappa_0}\,\xi_{T,2}^{(2)}}$$

On vérifie bien que :

$$\left[\frac{\kappa\beta}{nV_0}\right] = \left[\frac{\xi_{T,1}^{(2)}}{V_0}\right] \ :\ \left[\frac{\xi_{T,1}^{(2)}}{V_0}\right] = \frac{\kappa_0}{nk_BT V_0} = \frac{\kappa_0\beta}{nV_0}$$

$$\left[\frac{\mu\beta}{nV_0}\right] = \left[\frac{\xi_{T,2}^{(2)}}{V_0}\right] \ :\ \left[\frac{\xi_{T,2}^{(2)}}{V_0}\right] = \frac{\kappa_0}{n^2k_BV_0} = \frac{\kappa_0\beta T}{n^2V_0} = \underbrace{\frac{T\kappa_0}{n}}_{=[\mu]}\frac{\beta}{nV_0}$$

On vérifie aussi les dimensions de C_n et C_T :

$$[C_T] = \left[\frac{\kappa\beta}{nV_0}\right] = \frac{\kappa_0}{nV_0k_BT} = \frac{k_B}{m}\eta_0\frac{1}{nk_BT}\frac{\eta_0}{nk_BT} = \frac{\eta_0^2}{(nk_BT)^2}\frac{k_B}{m} = \frac{\cancel{n}\cancel{k_B}T}{\sigma^{d-1}}\frac{1}{(n\cancel{k_B}T)^2}\frac{\cancel{k_B}}{\cancel{m}} \propto n^{-2}T^{-1}$$

$$[C_n] = \left[\frac{\mu\beta}{nV_0}\right] = \frac{T}{n}\frac{\kappa_0}{nV_0k_BT} \propto \frac{T}{n}n^{-2}T^{-1} \propto n^{-3}T^0$$

Pour les dimensions de $\xi_{n,1}^{(2)}$ et $\xi_{n,2}^{(2)}$:

$$\left[\xi_{n,1}^{(2)}\frac{m\beta}{V_0}\right] = \left[\frac{k\beta}{nV_0}\right] \Rightarrow [\xi_{n,1}^{(2)}] = \left[\frac{\kappa_0\cancel{\beta}}{n\cancel{V_0}}\frac{\cancel{V_0}}{m\cancel{\beta}}\right] = \left[\frac{\kappa_0}{nm}\right]$$

$$\Rightarrow \boxed{\xi_{n,1}^{(2)*} = \frac{d}{d-1}\frac{nm}{\kappa_0}\xi_{n,1}^{(2)}}$$

$$\left[\xi_{n,2}^{(2)}\frac{m\beta}{V_0}\right] = \left[\frac{\mu\beta}{nV_0}\right] \Rightarrow [\xi_{n,2}^{(2)}] = \left[\frac{\cancel{V_0}}{m\cancel{\beta}}\frac{\mu\cancel{\beta}}{n\cancel{V_0}}\right] = \left[\frac{1}{nm}\frac{T\kappa_0}{n}\right] = \left[\frac{T\kappa_0}{n^2m}\right]$$

$$\Rightarrow \boxed{\xi_{n,2}^{(2)*} = \frac{d}{d-1}\frac{n^2m}{T\kappa_0}\xi_{n,2}^{(2)}}$$

Résolution : par les différents modèles :

1) calcul de V_ξ^* } mieux : cf Maxwell ?
2) calcul de $\xi_{A,i}^{(2)}$ en fonction de $C_n^{(2)}$ et $C_T^{(2)}$ ↙ on adimentionalise ici : calcule et résoud pour $C_n^{(2)*}$, $C_T^{(2)*}$
3) Insertion de ces relations obtenues sous 2) dans les Eqs. → résolution pour $C_n^{(2)}$ et $C_T^{(2)}$
4) Insertion des $C_n^{(2)}$ et $C_T^{(2)}$ dans $\xi_{A,i}^{(2)}$ et adimentionalisation

Modèle de Maxwell

1) Calcul de V_q^*

$$V_q^* = \frac{m^2\beta^2}{d(d+2)\nu_0 n}\int_{\mathbb{R}^d} dv\, v^4\, J\left[S_2(c^2)\mathcal{M}(v)\right] \qquad ; \mathcal{M}(v) = \frac{n}{V_T^d}\frac{1}{\pi^{d/2}}e^{-V^2/V_T^2}$$

$$S_2(c^2) = \frac{1}{2}c^4 - \frac{d+2}{2}c^2 + \frac{d(d+2)}{8}$$

$$\begin{cases} J[g] = p\,L_a g + (1-p)L_c g \\ L_a g = -J_a[f^{(0)},g] - J_a[g,f^{(0)}] \\ L_c g = -J_c[f^{(0)},g] - J_c[g,f^{(0)}] \end{cases}$$

$$\int_{\mathbb{R}^d} dv_1\, Y(v_1)\, L_a[\mathcal{M}X] = \sigma^{d-1}\phi(x)V_T^{1-x}\int_{\mathbb{R}^{2d}} dv_1 dv_2\, V_{12}^x f^{(0)}(V_1)\mathcal{M}(V_2)X(V_2)\left[Y(V_1)+Y(V_2)\right]$$

$$\int_{\mathbb{R}^d} dv_1\, Y(v_1)\, L_c[\mathcal{M}X] = -\sigma^{d-1}\frac{\phi(x)V_T^{1-x}}{S_d}\int_{\mathbb{R}^{2d}} dv_1 dv_2\, V_{12}^x f^{(0)}(V_1)\mathcal{M}(V_2)X(V_2)\int d\hat{\sigma}\,(b-1)\left[Y(V_1)+Y(V_2)\right]$$

<u>Ici</u>: $x=0$; $Y(V_1) = V_1^4$; $X(c) = S_2(c^2)$; $b\,\underline{v_i} = \underline{v_i} \mp (v_{12}\cdot\hat{\sigma})\hat{\sigma}$

L_a:

$$\int_{\mathbb{R}^d} dv_1\, V_1^4 L_a[\mathcal{M}S_2] = \sigma^{d-1}\phi V_T\int_{\mathbb{R}^{2d}} dv_1 dv_2\, f^{(0)}(V_1)\mathcal{M}(V_2)\, S_2(V_2^2)\left[V_1^4+V_2^4\right]$$

$$= \sigma^{d-1}\phi V_T\int_{\mathbb{R}^{2d}} dv_1 dv_2\,\frac{n}{V_T^d}\frac{1}{\pi^{d/2}}e^{-V_1^2/V_T^2}\frac{n}{V_T^d}\frac{1}{\pi^{d/2}}e^{-V_2^2/V_T^2}\, S_2(c_2^2)\left(V_1^4+V_2^4\right)$$

$$= \sigma^{d-1}\phi V_T\frac{n^2}{\cancel{V_T^{2d}}}\frac{V_T^4}{\pi^d}\int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2}e^{-c_2^2}\,\cancel{V_T^{2d}}\left(\frac{1}{2}c_2^4 - \frac{d+2}{2}c_2^2 + \frac{d(d+2)}{8}\right)\left(c_1^4+c_2^4\right)$$

$$= \sigma^{d-1}\phi V_T\, n^2\frac{V_T^4}{\pi^d}\Bigg[\int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2}c_1^4\left(\frac{1}{2}c_2^4 - \frac{d+2}{2}c_2^2 + \frac{d(d+2)}{8}\right)e^{-c_2^2}$$
$$+\int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2}e^{-c_2^2}\left(\frac{1}{2}c_2^8 - \frac{d+2}{2}c_2^6 + \frac{d(d+2)}{8}c_2^4\right)\Bigg]$$

$$= \sigma^{d-1}\phi V_T n^2 V_T^4\Bigg[\frac{\Gamma(\frac{d+4}{2})}{\Gamma(d/2)}\left\{\frac{1}{2}\frac{\Gamma(\frac{d+4}{2})}{\Gamma(d/2)} - \frac{d+2}{2}\frac{\Gamma(\frac{d+2}{2})}{\Gamma(d/2)} + \frac{d(d+2)}{8}\right\}$$
$$+\frac{1}{2}\frac{\Gamma(\frac{d+8}{2})}{\Gamma(d/2)} - \frac{d+2}{2}\frac{\Gamma(\frac{d+6}{2})}{\Gamma(d/2)} + \frac{d(d+2)}{8}\frac{\Gamma(\frac{d+4}{2})}{\Gamma(d/2)}\Bigg\}\Bigg]$$

$$= \sigma^{d-1}\phi V_T n^2 V_T^4\Bigg[\boxed{\frac{d+2}{2}\frac{d}{2}}\left\{\frac{1}{2}\frac{d+2}{2}\frac{d}{2} - \frac{d+2}{2}\frac{d}{2} + \frac{d(d+2)}{8}\right)$$
$$+\frac{1}{2}\frac{d+6}{2}\frac{d+4}{2}\boxed{\frac{d+2}{2}\frac{d}{2}} - \frac{d+2}{2}\frac{d+4}{2}\boxed{\frac{d+2}{2}\frac{d}{2}} + \frac{d(d+2)}{8}\boxed{\frac{d+2}{2}\frac{d}{2}}\Bigg]$$

$$= \sigma^{d-1}\phi V_T n^2 V_T^4\;\frac{d+2}{2}\frac{d}{2}\frac{1}{8}\left[d\cancel{(d+2)} - 2d\cancel{(d+2)} + d\cancel{(d+2)} + (d+6)(d+4) - (d+2)(d+4)2 + d(d+2)\right]$$

$$= \sigma^{d-1}\phi V_T n^2 V_T^4\,\frac{d(d+2)}{32}\left[\cancel{d^2}+\cancel{4d}+\cancel{6d}+24-\cancel{2d^2}-\cancel{8d}-\cancel{4d}-16+\cancel{d^2}+\cancel{2d}\right]$$

$$= \sigma^{d-1}\phi n^2 V_T^5\frac{d(d+2)}{32}\,8$$

$$= n^2\sigma^{d-1}\phi V_T^5\frac{d(d+2)}{4}$$

L_c:

$$\int_{\mathbb{R}^d} dv_1\, V_1^4 L_c[\mathcal{M}S_2] = -\sigma^{d-1}\frac{\phi V_T}{S_d}\int_{\mathbb{R}^{2d}} dv_1 dv_2\, f^{(0)}(V_1)\mathcal{M}(V_2)\, S_2(c_2^2)\int d\hat{\sigma}\,(b-1)\left[V_1^4+V_2^4\right]$$

$$bV_1^4 = \left[v_1 - (g\cdot\hat{\sigma})\hat{\sigma}\right]^4 = \left[V_1^2 + (g\cdot\hat{\sigma})^2 - 2(g\cdot\hat{\sigma})(V_1\cdot\hat{\sigma})\right]^2$$

$$= V_1^4 + (g\cdot\hat{\sigma})^4 + 2V_1^2(g\cdot\hat{\sigma})^2 + 4(g\cdot\hat{\sigma})^2(V_1\cdot\hat{\sigma})^2 - 4(g\cdot\hat{\sigma})(V_1\cdot\hat{\sigma})V_1^2 - 4(g\cdot\hat{\sigma})(V_1\cdot\hat{\sigma})(g\cdot\hat{\sigma})^2$$

$$= V_1^4 + (g\cdot\hat{\sigma})^4 + 4(g\cdot\hat{\sigma})^2(V_1\cdot\hat{\sigma})^2 + 2V_1^2(g\cdot\hat{\sigma})^2 - 4(g\cdot\hat{\sigma})(V_1\cdot\hat{\sigma})V_1^2 - 4(g\cdot\hat{\sigma})^3(V_1\cdot\hat{\sigma})$$

$$bV_2^4 = \left[v_2 + (g\cdot\hat{\sigma})\hat{\sigma}\right]^4 = V_2^4 + (g\cdot\hat{\sigma})^4 + 4(g\cdot\hat{\sigma})^2(V_2\cdot\hat{\sigma})^2 + 2V_2^2(g\cdot\hat{\sigma})^2 + 4(g\cdot\hat{\sigma})(V_2\cdot\hat{\sigma})V_2^2 + 4(g\cdot\hat{\sigma})^3(V_2\cdot\hat{\sigma})$$

$$(b-1)[V_1^4+V_2^4] = \cancel{V_1^4}+\cancel{V_2^4} + 2(g\cdot\hat\sigma)^4 + 4(g\cdot\hat\sigma)^2(V_1\cdot\hat\sigma)^2 + 4(g\cdot\hat\sigma)^2(V_2\cdot\hat\sigma)^2 + 2V_1^2(g\hat\sigma)^2 + 2V_2^2(g\hat\sigma)^2$$

$$-4(g\cdot\hat\sigma)(V_1\cdot\hat\sigma)V_1^2 + 4(g\cdot\hat\sigma)(V_2\cdot\hat\sigma)V_2^2 - 4(g\cdot\hat\sigma)^3(V_1\cdot\hat\sigma) + 4(g\cdot\hat\sigma)^3(V_2\cdot\hat\sigma)$$

$$-\cancel{V_1^4}-\cancel{V_2^4}$$

$$= 2(g\cdot\hat\sigma)^4 + (g\cdot\hat\sigma)^3[-4(V_1\cdot\hat\sigma)+4(V_2\cdot\hat\sigma)] + (g\cdot\hat\sigma)^2[4(V_1\cdot\hat\sigma)^2+4(V_2\cdot\hat\sigma)^2+2V_1^2+2V_2^2]$$

$$+(g\cdot\hat\sigma)[-4V_1^2(V_1\cdot\hat\sigma)+4V_2^2(V_2\cdot\hat\sigma)]$$

$$= 2(g\hat\sigma)^4 + (g\cdot\hat\sigma)^3(+4)(-V_{1i}+V_{2i})\sigma_i + (g\cdot\hat\sigma)^2\, 2[2V_{1i}V_{1j}\sigma_i\sigma_j + 2V_{2i}V_{2j}\sigma_i\sigma_j + V_1^2+V_2^2]$$

$$+(g\cdot\hat\sigma)\,4[-V_1^2V_{1i}+V_2^2V_{2i}]\sigma_i$$

$$= 2(g\cdot\hat\sigma)^4 + 4(g\cdot\hat\sigma)^3(V_{2i}-V_{1i})\sigma_i + 4(g\cdot\hat\sigma)^2(V_{1i}V_{1j}+V_{2i}V_{2j})\sigma_i\sigma_j + 2(g\cdot\hat\sigma)^2(V_1^2+V_2^2)$$

$$+4(g\cdot\hat\sigma)(-V_1^2V_{1i}+V_2^2V_{2i})\sigma_i$$

$$\int d\hat\sigma\,(b-1)(V_1^4+V_2^4) = 2\beta_4 g^4 + 4(V_{2i}-V_{1i})\beta_4 g^2 g_i + 4(V_{1i}V_{1j}+V_{2i}V_{2j})\frac{\beta_2}{d+2}g^0(2g_ig_j+g^2\delta_{ij}) + 2(V_1^2+V_2^2)g^2\beta_2$$

$$+4(V_2^2V_{2i}-V_1^2V_{1i})\beta_2 g^{1-1}g_i \qquad ;\ \underline{g}=\underline{V_1}-\underline{V_2}$$

$$= 2\beta_4(V_1^2+V_2^2-2V_1\cdot V_2)^2 + 4\beta_4(V_{2i}-V_{1i})(V_{1i}-V_{2i})(V_1^2+V_2^2-2V_1\cdot V_2)$$

$$+\frac{4\beta_2}{d+2}(V_{1i}V_{1j}+V_{2i}V_{2j})\left[2(V_{1i}-V_{2i})(V_{1j}-V_{2j})+(V_1^2+V_2^2-2V_1\cdot V_2)\delta_{ij}\right] + 2\beta_2(V_1^2+V_2^2)(V_1^2+V_2^2-\overset{\text{antisym.}}{\cancelto{0}{2V_1\cdot V_2}})$$

$$+4\beta_2(V_2^2V_{2i}-V_1^2V_{1i})(V_{1i}-V_{2i})$$

$$= 2\beta_4\Big[(V_1^2+V_2^2)^2 + \underset{\to\frac{4}{d}V_1^2V_2^2}{\boxed{4(V_1\cdot V_2)^2}} - 4(\cancelto{0:\text{antisym}}{V_1\cdot V_2})(V_1^2+V_2^2)\Big] + 4\beta_4(V_1^2+V_2^2-2V_1\cdot V_2)(V_{2i}V_{1i}-V_{2i}V_{2i}-V_{1i}V_{1i}+V_{1i}V_{2i})$$

$$+\frac{4\beta_2}{d+2}(V_1^2+V_2^2-2V_1\cdot V_2)(V_{1i}V_{1i}+V_{2i}V_{2i}) + \frac{4\beta_2}{d+2}\,2(V_{1i}V_{1j}+V_{2i}V_{2j})(V_{1i}V_{1j}-V_{1i}V_{2j}-V_{2i}V_{1j}+V_{2i}V_{2j})$$

$$+2\beta_2(V_1^4+V_2^4+2V_1^2V_2^2) + 4\beta_2(V_2^2V_{2i}V_{1i}-V_2^2V_{2i}V_{2i}-V_1^2V_{1i}V_{1i}+V_1^2V_{1i}V_{2i})$$

$$= 2\beta_4\left[V_1^4+V_2^4+2V_1^2V_2^2+\frac{4}{d}V_1^2V_2^2\right] + 4\beta_4(V_1^2+V_2^2-2V_1\cdot V_2)(2V_1\cdot V_2-V_1^2-V_2^2)$$

$$+\frac{4\beta_2}{d+2}(V_1^2+V_2^2-\cancelto{0:\text{antisym}}{2V_1\cdot V_2})(V_1^2+V_2^2)$$

$$+\frac{8\beta_2}{d+2}\Big[\overbrace{V_{1i}V_{1j}V_{1i}V_{1j}}^{=V_1^4} - \overbrace{V_{1i}V_{1j}V_{1i}V_{2j}}^{=V_1^2(V_1\cdot V_2)\to 0} - \overbrace{V_{1i}V_{1j}V_{2i}V_{1j}}^{=V_1^2(V_1\cdot V_2)\to 0} + \overbrace{V_{1i}V_{1j}V_{2i}V_{2j}}^{=(V_1\cdot V_2)^2}$$

$$+\underbrace{V_{2i}V_{2j}V_{1i}V_{1j}}_{=(V_1\cdot V_2)^2} - \underbrace{V_{2i}V_{2j}V_{1i}V_{2j}}_{=V_2^2(V_1\cdot V_2)\to 0} - \underbrace{V_{2i}V_{2j}V_{2i}V_{1j}}_{=V_2^2(V_1\cdot V_2)\to 0} + \underbrace{V_{2i}V_{2j}V_{2i}V_{2j}}_{=V_2^4}\Big]$$

$$+2\beta_2(V_1^4+V_2^4+2V_1^2V_2^2) + 4\beta_2\Big[V_2^2(\cancelto{0}{V_1\cdot V_2}) - V_2^2V_2^2 - V_1^2V_1^2 + V_1^2(\cancelto{0}{V_1\cdot V_2})\Big]$$

$$= 2\beta_4\left[V_1^4+V_2^4+\frac{2d+4}{d}V_1^2V_2^2\right] - 4\beta_4\Big[(V_1^2+V_2^2)^2 + \overset{\to\frac{4}{d}V_1^2V_2^2}{\boxed{4(V_1\cdot V_2)^2}} - 4(\cancelto{0}{V_1\cdot V_2})(V_1^2+V_2^2)\Big]$$

$$+\frac{4\beta_2}{d+2}\left[V_1^4+V_2^4+2V_1^2V_2^2\right] + \frac{8\beta_2}{d+2}\Big[V_1^4+V_2^4+\underset{\to\frac{2}{d}V_1^2V_2^2}{\boxed{2(V_1\cdot V_2)^2}}\Big] + 2\beta_2\left[V_1^4+V_2^4+2V_1^2V_2^2\right]$$

$$-4\beta_2\left[V_1^4+V_2^4\right] \qquad = \frac{2(d+2)}{d}V_1^2V_2^2$$

$$= 2\beta_4\left(V_1^4+V_2^4+\frac{2(d+2)}{d}V_1^2V_2^2\right) - 4\beta_4\Big(V_1^4+V_2^4+\boxed{2V_1^2V_2^2+\frac{4}{d}V_1^2V_2^2}\Big) + \frac{4\beta_2}{d+2}\left(V_1^4+V_2^4+2V_1^2V_2^2\right)$$

$$+\frac{8\beta_2}{d+2}\left(V_1^4+V_2^4+\frac{2}{d}V_1^2V_2^2\right) + 2\beta_2\left(V_1^4+V_2^4+2V_1^2V_2^2\right) - 4\beta_2\left(V_1^4+V_2^4\right)$$

$$= 2\beta_4\left[V_1^4+V_2^4+\frac{2(d+2)}{d}V_1^2V_2^2-2V_1^4-2V_2^4-2\frac{2(d+2)}{d}V_1^2V_2^2\right]$$

$$+\frac{4\beta_2}{d+2}\left[V_1^4+V_2^4+2V_1^2V_2^2+2V_1^4+2V_2^4+\frac{4}{d}V_1^2V_2^2\right]$$

$$+2\beta_2\left[V_1^4+V_2^4+2V_1^2V_2^2-2V_1^4-2V_2^4\right]$$

$$= 2\beta_4\left[-V_1^4-V_2^4-\frac{2(d+2)}{d}V_1^2V_2^2\right] + \frac{4\beta_2}{d+2}\left[3V_1^4+3V_2^4+\frac{2(d+2)}{d}V_1^2V_2^2\right] + 2\beta_2\left[-V_1^4-V_2^4+2V_1^2V_2^2\right]$$

$$= -2\beta_4\left[V_1^4+V_2^4+\frac{2(d+2)}{d}V_1^2V_2^2\right] + \frac{4\beta_2}{d+2}\left[3V_1^4+3V_2^4+\frac{2(d+2)}{d}V_1^2V_2^2-\frac{2}{d+2}V_1^4-\frac{2}{d+2}V_2^4+\frac{4}{d+2}V_1^2V_2^2\right]$$

$$= -2\beta_4\left[V_1^4+V_2^4+\frac{2(d+2)}{d}V_1^2V_2^2\right] + \frac{4\beta_2}{d+2}\left[V_1^4\frac{3(d+2)-2}{d+2} + V_2^4\frac{3(d+2)-2}{d+2} + V_1^2V_2^2\frac{2(d+2)^2+4d}{d(d+2)}\right]$$

$$\downarrow = 2\pi^{(d-1)/2}\frac{\Gamma(\frac{1+4}{2})}{\Gamma(\frac{d+4}{2})} = 2\pi^{(d-1)/2}\frac{1+2}{2}\frac{\Gamma(\frac{1+2}{2})}{\Gamma(\frac{d+2}{2})\frac{d+2}{2}} = \frac{3}{2}\frac{2}{d+2}\beta_2$$

$$= -\frac{4\beta_2}{d+2}\,\frac{3}{2}\left[V_1^4+V_2^4+\frac{2(d+2)}{d}V_1^2V_2^2\right] + \frac{4\beta_2}{d+2}\left[\ \dots\ \right]$$

$$= \frac{4\beta_2}{d+2}\left[V_1^4\frac{6(d+2)-4-3(d+2)}{2(d+2)} + V_2^4\frac{6(d+2)-4-3(d+2)}{2(d+2)} + V_1^2V_2^2\frac{4(d+2)^2+8d-6(d+2)^2}{2d(d+2)}\right]$$

$$= \frac{2\beta_2}{(d+2)^2}\left[V_1^4\left\{3(d+2)-4\right\} + V_2^4\left\{3(d+2)-4\right\} + V_1^2V_2^2\frac{8d-2(d+2)^2}{d}\right]$$

Así:

$$\int_{\mathbb{R}^d} dv\, V_1^4 L_c[MS_2] = -\sigma^{d-1}\frac{\emptyset V_T}{S_d}\int_{\mathbb{R}^{2d}} dv_1 dv_2\, \frac{n^2}{V_T^{2d}}\frac{1}{\pi^d} e^{-c_1^2}e^{-c_2^2}\left[\frac{1}{2}\frac{V_2^4}{V_T^4} - \frac{d+2}{2}\frac{V_2^2}{V_T^2} + \frac{d(d+2)}{8}\right]\frac{2\beta_2}{(d+2)^2}\left[\ \dots\ \right]$$

$$= -\sigma^{d-1}\frac{\emptyset V_T}{S_d}\frac{n^2}{\pi^d}V_T^4\int_{\mathbb{R}^{2d}} dc_1dc_2\, e^{-c_1^2}e^{-c_2^2}\frac{2\beta_2}{(d+2)^2}\left[\frac{1}{2}c_2^4 - \frac{d+2}{2}c_2^2 + \frac{d(d+2)}{8}\right]\left[c_1^4(3d-2) + c_2^4(3d-2) + c_1^2c_2^2\frac{8d-2(d+2)^2}{d}\right]$$

$$= -\sigma^{d-1}\frac{\emptyset V_T}{S_d}\frac{n^2}{\pi^d}V_T^4\frac{2\beta_2}{(d+2)^2}\int_{\mathbb{R}^{2d}} dc_1dc_2\, e^{-c_1^2-c_2^2}\Bigg[\frac{3d-2}{2}c_1^4c_2^4 + \frac{3d-2}{2}c_2^8 + \frac{8d-2(d+2)^2}{2d}c_1^2c_2^6 - \frac{(d+2)(3d-2)}{2}c_1^4c_2^2$$
$$- \frac{(d+2)(3d-2)}{2}c_2^6 - \frac{(d+2)}{2}\frac{8d-2(d+2)^2}{d}c_1^2c_2^4 + \boxed{\frac{d(d+2)(3d-2)}{8}c_1^4}$$
$$+ \boxed{\frac{d(d+2)(3d-2)}{8}c_2^4} + \frac{d(d+2)}{8}\frac{8d-2(d+2)^2}{d}c_1^2c_2^2\Bigg]$$

$$= -\sigma^{d-1}\frac{\emptyset V_T}{S_d}\frac{n^2}{\pi^d}V_T^4\frac{2\beta_2}{(d+2)^2}\pi^d\Bigg[\frac{3d-2}{2}\frac{d+2}{2}\frac{d}{2}\frac{d+2}{2}\frac{d}{2} + \frac{3d-2}{2}\frac{d+6}{2}\frac{d+4}{2}\frac{d+2}{2}\frac{d}{2} + \frac{8d-\cancel{2}(d+2)^2}{2d}\frac{d}{2}\frac{d+4}{2}\frac{d+2}{2}\frac{d}{2}$$
$$- \frac{(d+2)(3d-2)}{2}\frac{d+2}{2}\frac{d}{2}\frac{d}{2} - \frac{(d+2)(3d-2)}{2}\frac{d+4}{2}\frac{d+2}{2}\frac{d}{2} - \frac{d+2}{2}\frac{8d-2(d+2)^2}{\cancel{d}}\frac{\cancel{d}}{2}\frac{d+2}{2}\frac{d}{2}$$
$$+ \frac{d(d+2)(3d-2)}{4}\frac{d+2}{2}\frac{d}{2} + \frac{d(d+2)}{\cancel{8}4}\frac{1}{2}\frac{8d-2(d+2)^2}{d}\frac{d}{2}\frac{d}{2}\Bigg]$$

$$= -\sigma^{d-1}\frac{\emptyset V_T}{S_d}\frac{n^2}{\cancel{\pi^d}}V_T^4\frac{\cancel{2}\beta_2}{(d+2)^{\cancel{2}}}\cancel{\pi^d}\frac{\cancel{d+2}}{\cancel{2}}\frac{d}{2}\frac{1}{4}\Bigg[\frac{3d-2}{2}d(d+2) + \frac{3d-2}{2}(d+6)(d+4) + \frac{8d-2(d+2)^2}{2d}d(d+4)$$
$$- d(d+2)\cancel{(3d-2)} - (d+2)(d+4)(3d-2) - (d+2)\left[8d-2(d+2)^2\right]$$
$$+ d(d+2)\cancel{(3d-2)} + \frac{8d-2(d+2)^2}{2d}d^2\Bigg]$$

$$= -\sigma^{d-1}\frac{\emptyset V_T}{S_d}n^2V_T^4\frac{\beta_2}{d+2}\frac{d}{8}\Bigg[\frac{3d-2}{2}\left\{d(d+2)+(d+6)(d+4)\right\} + \frac{\cancel{8d}-2(d+2)^2}{\cancel{d}}\cancel{d}(d+2) - (d+2)(d+4)(3d-2)$$
$$- (d+2)\left[\cancel{8d}-2(d+2)^2\right]\Bigg]$$

$$= -\sigma^{d-1}\frac{\emptyset V_T}{S_d}n^2V_T^4\frac{\beta_2}{d+2}\frac{d}{8}\,\frac{3d-2}{2}\Big[\underbrace{d(d+2)+(d+6)(d+4) - 2(d+4)(d+2)}_{=\cancel{d^2}+\cancel{2d}+\cancel{d^2}+\cancel{4d}+\cancel{6d}+24-\cancel{2d^2}-\cancel{4d}-\cancel{8d}-16\ =8}\Big]$$

$$= -\sigma^{d-1}\frac{\emptyset V_T}{S_d}n^2V_T^4\frac{\beta_2}{d+2}\frac{d(3d-2)}{2}\quad ;\ \beta_2 = 2\pi^{(d-1)/2}\frac{\Gamma(\frac{2+1}{2})}{\Gamma(\frac{d+2}{2})}\quad ;\ S_d = 2\pi^{d/2}/\Gamma(d/2)$$

$$= -\sigma^{d-1}\emptyset V_T^5 n^2\,\cancel{2}\pi^{(d-1)/2}\frac{\Gamma(\frac{2+1}{2})}{\Gamma(\frac{d+2}{2})}\frac{\Gamma(d/2)}{\cancel{2}\pi^{d/2}}\frac{d(3d-2)}{2(d+2)}$$

$$= -n^2\sigma^{d-1}\emptyset V_T^5\frac{\cancel{d}(3d-2)}{\cancel{2}(d+2)}\cancel{\pi^{-1/2}}\,\frac{1}{2}\underbrace{\cancel{\Gamma(1/2)}}_{=\cancel{\sqrt{\pi}}}\frac{\cancel{\Gamma(d/2)}}{\cancel{\frac{d}{2}}\cancel{\Gamma(d/2)}}$$

$$= -n^2\sigma^{d-1}\emptyset V_T^5\frac{3d-2}{2(d+2)}$$

Conclusión:

$$V_q^* = \frac{m^2\beta^2}{d(d+2)V_0 n}\left[p\, n^2\sigma^{d-1}\emptyset V_T^5\frac{d(d+2)}{4} + (1-p)(-1)\, n^2\sigma^{d-1}\emptyset V_T^5\frac{3d-2}{2(d+2)}\right]$$

$$\boxed{V_q^* = \frac{m^2\beta^2}{d(d+2)V_0\cancel{n}}n^{\cancel{2}}\sigma^{d-1}\emptyset V_T^5\left[p\frac{d(d+2)}{4} - (1-p)\frac{3d-2}{2(d+2)}\right]}$$

$$\xi^{(2)}_{n,1} = \frac{2}{n}\,\omega[f^{(0)}, S_2 M]\, C_T^{(2)} \qquad ; \omega[f,g] = -\int_{\mathbb{R}^d} dv_1\, J_a[f,g]$$

$$= \frac{2}{n}\sigma^{d-1}\varnothing V_T C_T^{(2)} \frac{n^2}{V_T^{2d}}\frac{1}{\pi^d}\int_{\mathbb{R}^d} dv_1\, e^{-V_1^2/V_T^2}\int_{\mathbb{R}^d} dv_2\, S_2\, e^{-V_2^2/V_T^2} \qquad = \sigma^{d-1}\varnothing V_T \int_{\mathbb{R}^d} dv_1\, g(r,v_1;t)\int_{\mathbb{R}^d} dv_2\, f(r,v_2,t)$$

$$= \frac{2}{\cancel{n}}\sigma^{d-1}\varnothing V_T C_T^{(2)} \frac{\cancel{n^2}}{\cancel{V_T^{2d}}}\cancel{V_T}\frac{\pi^{d/2}}{\pi^d}\cancel{V_T^d}\int_{\mathbb{R}^d} dc_2\, e^{-c_2^2}\left[\frac{1}{2}c_2^4 - \frac{d+2}{2}c_2^2 + \frac{d(d+2)}{8}\right]$$

$$= 2\sigma^{d-1}\varnothing V_T C_T^{(2)}\left[\frac{1}{2}\frac{d+2}{2}\frac{d}{2} - \frac{d+2}{2}\frac{d}{2} + \frac{d(d+2)}{2\cdot 4\cancel{8}}\right]$$

$$= 2\sigma^{d-1}\varnothing V_T C_T^{(2)}\frac{d(d+2)}{4}\,\frac{1}{2}\left[1 - 2 + 1\right]$$

$$= 0$$

⟶ parce que justement $a_2 = 0$!

$$\xi^{(2)}_{n,2} = \frac{2}{n}\,\omega[f^{(0)}, S_2 M]\, C_n^{(2)} = 0$$

$$\xi^{(2)}_{u_i,1} = \frac{1}{nV_T}\,\omega[f^{(0)}, V_i S_2 M]\, C_T^{(2)} + \frac{1}{nV_T}\,\omega[S_2 M, V_i f^{(0)}]\, C_T^{(2)}$$

$$= \frac{1}{nV_T}C_T^{(2)}\sigma^{d-1}\varnothing V_T\int_{\mathbb{R}^d} dv_1 \underbrace{V_{1i} S_2 M}_{\text{impair}\to 0}\int_{\mathbb{R}^d} dv_2\, f^{(0)}(V_2) + \frac{1}{nV_T}C_T^{(2)}\sigma^{d-1}\varnothing V_T\int_{\mathbb{R}^d} dv_1 \underbrace{V_{1i} f^{(0)}(V_1)}_{\text{impair}\to 0}\int_{\mathbb{R}^d} dv_2\, S_2 M$$

$$= 0$$

$$\xi^{(2)}_{u_i,2} = 0$$

$$\xi^{(2)}_{T,1} = \frac{m}{k_B T d}\,\omega[f^{(0)}, V^2 S_2 M]\, C_T^{(2)} + \frac{m}{k_B T d}\,\omega[S_2 M, V^2 f^{(0)}]\, C_T^{(2)} - \underbrace{\xi^{(2)}_{n,1}}_{=0}$$

$$= \frac{m}{k_B T d}C_T^{(2)}\sigma^{d-1}\varnothing V_T\int_{\mathbb{R}^d} dv_1\, V^2 S_2 M\int_{\mathbb{R}^d} dv_2\, f^{(0)} + \frac{m}{k_B T d}C_T^{(2)}\sigma^{d-1}\varnothing V_T\int_{\mathbb{R}^d} dv_1\, V^2 f^{(0)}\int_{\mathbb{R}^d} dv_2\, S_2 M$$

$$= \frac{m}{k_B T d}C_T^{(2)}\sigma^{d-1}\varnothing V_T\frac{n^2}{\cancel{V_T^{2d}}}\frac{1}{\pi^{d/2}}\frac{1}{\cancel{\pi^{d/2}}}V_T^2\int_{\mathbb{R}^d} dc\, c^2 e^{-c^2}\left[\frac{1}{2}c^4 - \frac{d+2}{2}c^2 + \frac{d(d+2)}{8}\right]\cancel{V_T^{2d}}$$

$$+ \frac{m}{k_B T d}C_T^{(2)}\sigma^{d-1}\varnothing V_T\frac{n^2}{\cancel{V_T^{2d}}}\frac{1}{\pi^{d/2}}\frac{1}{\pi^{d/2}}\cancel{V_T^{2d}}V_T^2\int_{\mathbb{R}^d} dc\, c^2 e^{-c^2}\underbrace{\int_{\mathbb{R}^d} dc\left[\frac{1}{2}c^4 - \frac{d+2}{2}c^2 + \frac{d(d+2)}{8}\right]e^{-c^2}}_{=\pi^{d/2}\frac{d(d+2)}{4}\frac{1}{2}[1-2+1] = 0}$$

$$= \frac{m}{k_B T d}C_T^{(2)}\sigma^{d-1}\varnothing V_T^3 n^2\left[\frac{1}{2}\frac{d+4}{2}\frac{d+2}{2}\frac{d}{2} - \frac{d+2}{2}\frac{d+2}{2}\frac{d}{2} + \frac{d(d+2)}{8}\frac{d}{2}\right]$$

$$= \frac{m}{k_B T d}C_T^{(2)}\sigma^{d-1}\varnothing V_T^3 n^2\frac{d(d+2)}{8}\,\frac{1}{2}\left[\underbrace{d+4 - 2(d+2)}_{=\cancel{d}+\cancel{4}-\cancel{2d}-\cancel{4}+\cancel{d}} + d\right]$$

$$= 0$$

$$\xi^{(2)}_{T,2} = 0$$

Conclusion :

ar le modèle de Maxwell, tous les taux de déclin à l'ordre 2 linéaire sont nuls.

ourquoi? $a_2 = 0$ ET pas de vitesse relative dans l'opérateur de collision

$\Rightarrow$ ces taux seront non nuls pour VHP. (prédiction)

Modèle VHP : $a_2 = 0$

2) Calcul de $\xi_{A,i}^{(2)}$:

$$\omega[f,g] = -\int_{\mathbb{R}^d} dv_1\, J_a[f,g] = \sigma^{d-1}\frac{\varnothing}{V_T}\int_{\mathbb{R}^d} dv_1\, g(r,v_1;t)\int_{\mathbb{R}^d} dv_2\, V_{12}^2\, f(r,v_2;t)$$

$$\xi_{n,1}^{(2)} = \frac{2}{n} C_T^{(2)}\,\omega[f^{(0)}, S_2\mathcal{M}]$$

$$= \frac{2}{n} C_T^{(2)}\,\sigma^{d-1}\frac{\varnothing}{V_T}\int_{\mathbb{R}^d} dv_1\, S_2\mathcal{M}(v_1)\int_{\mathbb{R}^d} dv_2\, f^{(0)}\, V_{12}^2$$

$$= \frac{2}{n} C_T^{(2)}\,\sigma^{d-1}\frac{\varnothing}{V_T}\frac{n^2}{V_T^{2d}}\frac{1}{\pi^d}\int_{\mathbb{R}^d} dv_1\, e^{-V_1^2/V_T^2}\left[\frac{1}{2}C_1^4 - \frac{d+2}{2}C_1^2 + \frac{d(d+2)}{8}\right]\int_{\mathbb{R}^d} dv_2\, (V_1^2+V_2^2-\overset{\text{impair}}{\cancel{2V_1\cdot V_2}})\, e^{-V_2^2/V_T^2}$$

$$= \frac{2}{\cancel{n}} C_T^{(2)}\,\sigma^{d-1}\frac{\varnothing}{V_T}\frac{n^{\cancel{2}}}{\cancel{V_T^{2d}}}\frac{1}{\pi^d}\cancel{V_T^{2d}}\int_{\mathbb{R}^d} dc_1\, e^{-c_1^2}\left[\frac{1}{2}c_1^4 - \frac{d+2}{2}c_1^2 + \frac{d(d+2)}{8}\right]\left[V_T^2 C_1^2 \underbrace{\int_{\mathbb{R}^d} dc_2\, e^{-c_2^2}}_{=\pi^{d/2}} + V_T^2 \underbrace{\int_{\mathbb{R}^d} dc_2\, c_2^2 e^{-c_2^2}}_{=\pi^{d/2}\frac{d}{2}}\right]$$

$$= 2C_T^{(2)}\sigma^{d-1}\frac{\varnothing}{V_T}\frac{n}{\pi^{d/2}}V_T^2\int_{\mathbb{R}^d} dc_1\, e^{-c_1^2}\left[\frac{1}{2}c_1^4 - \frac{d+2}{2}c_1^2 + \frac{d(d+2)}{8}\right]\left[c_1^2 + \underset{\substack{\downarrow\\ \text{donne zéro : cf. calcul de } \xi_{n,1}^{(2)} \text{ de Maxwell}}}{\frac{d}{2}}\right]$$

$$= 2C_T^{(2)}\sigma^{d-1}\frac{\varnothing}{\cancel{V_T}}\frac{n}{\pi^{d/2}}V_T^{\cancel{2}}\int_{\mathbb{R}^d} dc_1\, e^{-c_1^2}\left[\frac{1}{2}c_1^6 - \frac{d+2}{2}c_1^4 + \frac{d(d+2)}{8}c_1^2\right]$$

$$= 2C_T^{(2)}\sigma^{d-1}\varnothing\,\frac{nV_T}{\cancel{\pi^{d/2}}}\cancel{\pi^{d/2}}\left[\frac{1}{2}\,\frac{d+4}{2}\,\frac{d+2}{2}\,\frac{d}{2} - \frac{d+2}{2}\,\frac{d+2}{2}\,\frac{d}{2} + \frac{d(d+2)}{8}\,\frac{d}{2}\right]$$

$$= 2C_T^{(2)}\sigma^{d-1}\varnothing\, nV_T\,\frac{d(d+2)}{8}\,\frac{1}{2}\Big[\underbrace{d+4 \; -2(d+2) \; +d}_{=2d+4-2d-4=0}\Big]$$

$$= 0$$

$$\xi_{n,2}^{(2)} = 0$$

$$\xi_{a_i,1}^{(2)} = \frac{1}{nV_T}\omega[f^{(0)}, V_i S_2 M]\, C_T^{(2)} + \frac{1}{nV_T}\omega[S_2\mathcal{M}, V_i f^{(0)}]\, C_T^{(2)}$$

$$= \frac{1}{nV_T}C_T^{(2)}\sigma^{d-1}\frac{\varnothing}{V_T}\int_{\mathbb{R}^d} dv_1\, V_{1i}\, S_2\mathcal{M}\int_{\mathbb{R}^d} dv_2\, V_{12}^2 f^{(0)}(v_2) + \frac{1}{nV_T}C_T^{(2)}\sigma^{d-1}\frac{\varnothing}{V_T}\int_{\mathbb{R}^d} dv_1\, V_{1i} f^{(0)}\int_{\mathbb{R}^d} dv_2\, S_2\mathcal{M}\, V_{12}^2$$

$$= \frac{1}{nV_T}C_T^{(2)}\sigma^{d-1}\frac{\varnothing}{V_T}\frac{n^2}{V_T^{2d}}\frac{1}{\pi^d}\int_{\mathbb{R}^d} dv_1\, V_{1i}\, S_2\, e^{-V_1^2/V_T^2}\int_{\mathbb{R}^d} dv_2\, (\overset{0}{\cancel{V_1^2}}+\overset{0}{\cancel{V_2^2}}-2V_1\cdot V_2)\, e^{-V_2^2/V_T^2}$$

$$+ \frac{1}{nV_T}C_T^{(2)}\sigma^{d-1}\frac{\varnothing}{V_T}\frac{n^2}{V_T^{2d}}\frac{1}{\pi^d}\int_{\mathbb{R}^d} dv_1\, V_{1i}\, e^{-V_1^2/V_T^2}\int_{\mathbb{R}^d} dv_2\, (\overset{0}{\cancel{V_1^2}}+\overset{0}{\cancel{V_2^2}}-2V_1\cdot V_2)\, S_2\, e^{-V_2^2/V_T^2}$$

$$= (-2)\frac{1}{nV_T}C_T^{(2)}\sigma^{d-1}\frac{\varnothing}{V_T}\frac{n^2}{V_T^{2d}}\frac{1}{\pi^d}V_T^{2d}\,V_T^3\int_{\mathbb{R}^d} dc_1\, C_{1i}\, S_2 e^{-C_1^2}\overset{0}{\cancel{\int_{\mathbb{R}^d} dc_2\, C_{1j}\, \overset{\text{impair !}}{C_{2j}}\, e^{-C_2^2}}}$$

$$(-2)\frac{1}{nV_T}C_T^{(2)}\sigma^{d-1}\frac{\varnothing}{V_T}\frac{n^2}{V_T^{2d}}\frac{1}{\pi^d}V_T^{2d}\,V_T^3\int_{\mathbb{R}^d} dc_1\, C_{1i}\, e^{-C_1^2}\overset{0}{\cancel{\int_{\mathbb{R}^d} dc_2\, C_{1j}\, \overset{\text{impair !}}{C_{2j}}\, S_2 e^{-C_2^2}}}$$

$$= 0$$

$$\xi_{a_i,2}^{(2)} = 0$$

$$\xi_{T,1}^{(2)} = \frac{m}{nk_BTd}C_T^{(2)}\,\omega[f^{(0)}, V^2 S_2\mathcal{M}] + \frac{m}{nk_BTd}C_T^{(2)}\,\omega[S_2\mathcal{M}, V^2 f^{(0)}] - \underbrace{\xi_{n,1}^{(2)}}_{=0}$$

$$= \frac{m}{nk_BTd}C_T^{(2)}\sigma^{d-1}\frac{\varnothing}{V_T}\int_{\mathbb{R}^d} dv_1\, V_1^2 S_2\mathcal{M}\int_{\mathbb{R}^d} dv_2\, V_{12}^2 f^{(0)} + \frac{m}{nk_BTd}C_T^{(2)}\sigma^{d-1}\frac{\varnothing}{V_T}\int_{\mathbb{R}^d} dv_1\, V_1^2 f^{(0)}\int_{\mathbb{R}^d} dv_2\, V_{12}^2 S_2\mathcal{M}$$

$$= \frac{m}{nk_BTd}C_T^{(2)}\sigma^{d-1}\frac{\varnothing}{V_T}\frac{n^2}{\cancel{V_T^{2d}}}\frac{1}{\pi^d}\cancel{V_T^{2d}}\,V_T^4\int_{\mathbb{R}^d} dc_1\, C_1^2 S_2 e^{-C_1^2}\int_{\mathbb{R}^d} dc_2\, (C_1^2+C_2^2-\overset{0}{\cancel{2C_1\cdot C_2}})\, e^{-C_2^2}$$

$$+ \frac{m}{nk_BTd}C_T^{(2)}\sigma^{d-1}\frac{\varnothing}{V_T}\frac{n^2}{\cancel{V_T^{2d}}}\frac{1}{\pi^d}\cancel{V_T^{2d}}\,V_T^4\int_{\mathbb{R}^d} dc_1\, C_1^2 e^{-C_1^2}\int_{\mathbb{R}^d} dc_2\, (C_1^2+C_2^2-\overset{0}{\cancel{2C_1\cdot C_2}})\, S_2\, e^{-C_2^2}$$

$$= \frac{m}{nk_BTd} C_T^{(2)} \sigma^{d-1} \phi\, n^2 \frac{V_T^3}{\pi^d} \int_{\mathbb{R}^d} dc_1\, c_1^2 \left[\frac{1}{2}c_1^4 - \frac{d+2}{2}c_1^2 + \frac{d(d+2)}{8}\right] e^{-c_1^2} \int_{\mathbb{R}^d} dc_2\, \underbrace{(c_1^2+c_2^2)\,e^{-c_2^2}}_{=c_1^2\pi^{d/2}+\frac{d}{2}\pi^{d/2}}$$

$$+ \frac{m}{nk_BTd} C_T^{(2)} \sigma^{d-1} \phi\, n^2 \frac{V_T^3}{\pi^d} \int_{\mathbb{R}^d} dc_1\, c_1^2 e^{-c_1^2} \int_{\mathbb{R}^d} dc_2\, e^{-c_2^2} \left(\underset{\uparrow\ \text{donne contrib. nulle (cf p. 15)}}{c_1^2} + \overset{\text{donne contrib. nulle}\ \downarrow}{c_2^2}\right)\left[\frac{1}{2}c_2^4 - \frac{d+2}{2}c_2^2 + \frac{d(d+2)}{8}\right]$$

$$= \frac{m}{nk_BTd} C_T^{(2)} \sigma^{d-1} \phi\, n^2 \frac{V_T^3}{\pi^{d/2}} \int_{\mathbb{R}^d} dc_1 \left[\frac{1}{2}c_1^6 - \frac{d+2}{2}c_1^4 + \frac{d(d+2)}{8}c_1^2\right] e^{-c_1^2} \left(c_1^2 + \underset{\uparrow\ \text{donne contrib. nulle}}{\frac{d}{2}}\right)$$

$$= \frac{m}{nk_BTd} C_T^{(2)} \sigma^{d-1} \phi\, n^2 \frac{V_T^3}{\pi^{d/2}} \int_{\mathbb{R}^d} dc_1\, e^{-c_1^2} \left[\frac{1}{2}c_1^8 - \frac{d+2}{2}c_1^6 + \frac{d(d+2)}{8}c_1^4\right]$$

$$= \frac{m}{nk_BTd} C_T^{(2)} \sigma^{d-1} \phi\, n^2 V_T^3 \left[\frac{1}{2}\,\frac{d+6}{2}\,\frac{d+4}{2}\,\frac{d+2}{2}\,\frac{d}{2} - \frac{d+2}{2}\,\frac{d+4}{2}\,\frac{d+2}{2}\,\frac{d}{2} + \frac{d(d+2)}{8}\,\frac{d+2}{2}\,\frac{d}{2}\right]$$

$$= \frac{m}{nk_BTd} C_T^{(2)} \sigma^{d-1} \phi\, n^2 V_T^3\, \frac{d(d+2)}{16}\,\frac{1}{2}\Big[\underbrace{(d+6)(d+4) - (d+2)(d+4)2 + d(d+2)}\Big]$$

$$= \cancel{d^2} + \cancel{4d} + \cancel{6d} + 24 - \cancel{2d^2} - \cancel{8d} - \cancel{4d} - 16 + \cancel{d^2} + \cancel{2d}$$

$$= 8$$

$$\boxed{\xi_{T,1}^{(2)} = \frac{m}{k_BTd} C_T^{(2)} \sigma^{d-1} \phi\, n\, V_T^3 \frac{d(d+2)}{4}}$$

$$\boxed{\xi_{T,2}^{(2)} = \frac{m}{k_BTd} C_n^{(2)} \sigma^{d-1} \phi\, n\, V_T^3 \frac{d(d+2)}{4}}$$

$$= C_T^{(2)} \sigma^{d-1} \phi\, n V_T \frac{d(d+2)}{2}$$

✓

1) Calcul de V_ξ^* : similaire à p. 11

$\int dc\, e^{-c^2} c^k S_2(c^2) = 0\ ,\ k=0,2$; $\int dc\, e^{-c^2} c^4 S_2(c^2) = \pi^{d/2}\frac{d(d+2)}{4}$ (*) ✓

$\int dc\, e^{-c^2} c^6 S_2(c^2) = \pi^{d/2}\frac{3d(d+2)(d+4)}{8}$ ✓

$\int dc\, e^{-c^2} c^8 S_2(c^2) = \pi^{d/2}\frac{3d(d+2)(d+4)(d+6)}{8}$ ✓

$$V_\xi^* = \frac{m^2\beta^2}{d(d+2)V_0\, n} \int_{\mathbb{R}^d} dv\, v^4\, J\left[S_2(c^2)\mathcal{M}(v)\right]$$

avec:

$$J = p L_a + (1-p) L_c$$

$$\int_{\mathbb{R}^d} dv_1\, Y(v_1)\, L_a[\mathcal{M}X] = \sigma^{d-1}\phi(x)\, V_T^{1-x} \int_{\mathbb{R}^{2d}} dv_1 dv_2\, v_{12}^x f^{(0)}(v_1)\mathcal{M}(v_2)X(v_2)\left[Y(v_1) + Y(v_2)\right]$$

$$\int_{\mathbb{R}^d} dv_1\, Y(v_1)\, L_c[\mathcal{M}X] = -\sigma^{d-1}\frac{\phi(x)\, V_T^{1-x}}{S_d} \int_{\mathbb{R}^{2d}} dv_1 dv_2\, v_{12}^x f^{(0)}(v_1)\mathcal{M}(v_2)X(v_2)\int d\hat{\sigma}\,(b-1)\left[Y(v_1) + Y(v_2)\right]$$

Ici: $x=2$; $Y(v_1) = v_1^4$; $X(c) = S_2(c_2^2)$; $b\underline{v_i} = \underline{v_i} \mp (\underline{g}\cdot\hat{\sigma})\hat{\sigma}$; $\underline{g} = \underline{v_1} - \underline{v_2}$

$\underline{L_a}$:

$$\int_{\mathbb{R}^d} dv_1\, v_1^4\, L_a[\mathcal{M}S_2] = \sigma^{d-1}\frac{\phi}{V_T}\int_{\mathbb{R}^{2d}} dv_1 dv_2\, v_{12}^2 f^{(0)}(v_1)\mathcal{M}(v_2)\, S_2(v_2^2)\left[v_1^4 + v_2^4\right]$$

$$= \sigma^{d-1}\frac{\phi}{V_T}\,\frac{n^2}{\cancel{V_T^{2d}}}\,\frac{1}{\pi^d}\,\cancel{V_T^{2d}}\,V_T^6 \int_{\mathbb{R}^{2d}} dc_1 dc_2\, \underbrace{c_{12}^2}_{=c_1^2+c_2^2-2\cancel{c_1\cdot c_2}\to 0\,:\ \text{symétrie}} e^{-c_1^2}e^{-c_2^2}\left[\frac{1}{2}c_2^4 - \frac{d+2}{2}c_2^2 + \frac{d(d+2)}{8}\right]\left[c_1^4 + c_2^4\right]$$

$$= \sigma^{d-1}\phi\, n^2\frac{V_T^5}{\pi^d}\int_{\mathbb{R}^{2d}} dc_1 dc_2 \left[\frac{1}{2}c_2^4 - \frac{d+2}{2}c_2^2 + \frac{d(d+2)}{8}\right]\left[\underset{\xrightarrow{(*)}0}{c_1^6} + c_2^4 c_1^2 + \underset{\xrightarrow{(*)}0}{c_1^4 c_2^2} + c_2^6\right] e^{-c_1^2}e^{-c_2^2}$$

$$= \sigma^{d-1}\phi\, n^2\frac{V_T^5}{\pi^d}\left[\int_{\mathbb{R}^{2d}} dc_1 dc_2\, \underset{\downarrow\ \pi^{d/2}\frac{d}{2}}{c_1^2}\, e^{-c_1^2}e^{-c_2^2}\underbrace{c_2^4 S_2(c_2^2)}_{\to\pi^{d/2}\frac{d(d+2)}{4}} + \underbrace{\int_{\mathbb{R}^d} dc_1\, e^{-c_1^2}}_{\to\pi^{d/2}}\int_{\mathbb{R}^d} dc_2\, e^{-c_2^2} c_2^6 S_2(c_2^2)\right]$$

$$= \sigma^{d-1}\phi\, n^2 V_T^5\left[\frac{d}{2}\,\frac{d(d+2)}{4} + \frac{1}{\pi^{d/2}}\int_{\mathbb{R}^d} dc\, e^{-c^2}\left[\frac{1}{2}c^{10} - \frac{d+2}{2}c^8 + \frac{d(d+2)}{8}c^6\right]\right]$$

$$= \sigma^{d-1}\phi\, n^2 V_T^5\left[\frac{d^2(d+2)}{8} + \frac{1}{\pi^{d/2}}\,\cancel{\pi^{d/2}}\left\{\frac{1}{2}\,\frac{d+8}{2}\,\frac{d+6}{2}\,\frac{d+4}{2}\,\frac{d+2}{2}\,\frac{d}{2} - \frac{d+2}{2}\,\frac{d+6}{2}\,\frac{d+4}{2}\,\frac{d+2}{2}\,\frac{d}{2} + \frac{d(d+2)}{8}\,\frac{d+4}{2}\,\frac{d+2}{2}\,\frac{d}{2}\right\}\right]$$

$$= \sigma^{d-1}\phi\, n^2 V_T^5\left[\frac{d^2(d+2)}{8} + \frac{(d+4)(d+2)d}{32}\,\frac{1}{2}\Big\{\underbrace{(d+8)(d+6) - 2(d+2)(d+6) + d(d+2)}\Big\}\right]$$

$$= \cancel{d^2} + \cancel{6d} + \cancel{8d} + 48 - \cancel{2d^2} - 12d - \cancel{4d} - 24 + \cancel{d^2} + \cancel{2d}$$

$$= 24$$

$$= \sigma^{d-1} \varnothing n^2 V_T^5 \left[\frac{d^2(d+2)}{8} + \frac{3d(d+2)(d+4)}{8} \right]$$

$$= \sigma^{d-1} \varnothing n^2 V_T^5 \frac{d(d+2)}{8} \left[d + 3(d+4) \right]$$

$$= \sigma^{d-1} \varnothing n^2 V_T^5 \frac{d(d+2)}{8} \left[4d + 4\cdot 3 \right]$$

$$= \sigma^{d-1} \varnothing n^2 V_T^5 \frac{d(d+2)(d+3)}{2} \quad \checkmark_{\text{vérifié}}$$

<u>L_c</u> :

$$\int_{\mathbb{R}^d} dV_1 V_1^4 L_c[M S_2] = -\sigma^{d-1} \frac{\varnothing}{S_d V_T} \int_{\mathbb{R}^{2d}} dV_1 dV_2 \boxed{V_{12}^2} f^{(0)}(V_1) M(V_2) S_2(V_2^2) \int d\hat{\sigma}\, (b-1) \left[V_1^4 + V_2^4 \right]$$

$$= \underbrace{V_1^2+V_2^2}_{\text{pair}} - 2\underbrace{V_{1k}V_{2k}}_{\text{impair}}$$: on doit garder tous les termes de ↑

Cf. correction

$$b V_1^4 = \left[V_1 - (g\cdot\hat{\sigma})\hat{\sigma} \right]^4$$

$$= \left[V_1^2 + (g\cdot\hat{\sigma})^2 - 2(g\cdot\hat{\sigma})(V_1\cdot\hat{\sigma}) \right]^2$$

$$= (V_1^2 + (g\cdot\hat{\sigma})^2)^2 + 4(g\cdot\hat{\sigma})^2(V_1\cdot\hat{\sigma})^2 - 4(V_1^2 + (g\cdot\hat{\sigma})^2)(g\cdot\hat{\sigma})(V_1\cdot\hat{\sigma})$$

$$= V_1^4 + (g\cdot\hat{\sigma})^4 + 2V_1^2(g\cdot\hat{\sigma})^2 + 4(g\cdot\hat{\sigma})^2(V_1\cdot\hat{\sigma})^2 - 4V_1^2(g\cdot\hat{\sigma})(V_1\cdot\hat{\sigma}) - 4(g\cdot\hat{\sigma})^3(V_1\cdot\hat{\sigma})$$

$$= V_1^4 + (g\cdot\hat{\sigma})^4 - 4(g\cdot\hat{\sigma})^3(V_1\cdot\hat{\sigma}) + (g\cdot\hat{\sigma})^2\left[2V_1^2 + 4(V_1\cdot\hat{\sigma})^2\right] - 4(g\cdot\hat{\sigma})V_1^2(V_1\cdot\hat{\sigma})$$

$$b V_2^4 = V_2^4 + (g\cdot\hat{\sigma})^4 + 4(g\cdot\hat{\sigma})^3(V_2\cdot\hat{\sigma}) + (g\cdot\hat{\sigma})^2\left[2V_2^2 + 4(V_2\cdot\hat{\sigma})^2\right] + 4(g\cdot\hat{\sigma})V_2^2(V_2\cdot\hat{\sigma})$$

$$(b-1)\left[V_1^4+V_2^4\right] = 2(g\cdot\hat{\sigma})^4 + (g\cdot\hat{\sigma})^3\left[4V_{2i}\sigma_i - 4V_{1i}\sigma_i\right] + (g\cdot\hat{\sigma})^2\left[2V_1^2 + 2V_2^2 + 4V_{1i}V_{1j}\sigma_i\sigma_j + 4V_{2i}V_{2j}\sigma_i\sigma_j\right]$$
$$+ (g\cdot\hat{\sigma})\left[4V_2^2 V_{2i}\sigma_i - 4V_1^2 V_{1i}\sigma_i\right]$$

$$= 2(g\cdot\hat{\sigma})^{\overset{\checkmark}{4}} + 4(V_{2i}-V_{1i})(g\cdot\hat{\sigma})^3\overset{\checkmark}{\sigma_i} + 2(V_1^2+V_2^2)(\overset{\checkmark}{g}\cdot\hat{\sigma})^2 + 4(\overset{\checkmark}{V_{1i}}V_{1j} + V_{2i}V_{2j})(g\cdot\hat{\sigma})^2\sigma_i\sigma_j$$
$$+ 4(V_2^2V_{2i} - V_1^2V_{1i})(\overset{\checkmark}{g}\cdot\hat{\sigma})\sigma_i$$

$$\int d\hat{\sigma}\,(b-1)\left[V_1^4+V_2^4\right] = 2\beta_4 \underline{\underline{g}}^4 + 4(V_{2i}-V_{1i})\beta_4 \underline{\underline{g}}^2 g_i + 2(V_1^2+V_2^2)\beta_2 \underline{\underline{g}}^2 + 4(V_{1i}V_{1j}+V_{2i}V_{2j})\frac{\beta_2}{d+2} g^{ⓘ{2-2}}(2g_ig_j + \underline{\underline{g}}^2\delta_{ij})$$
$$+ 4(V_2^2V_{2i} - V_1^2V_{1i})\beta_2 g^{(1-1)} g_i \qquad ;\ g^2 = V_1^2 + V_2^2 - 2V_{1i}V_{2i}$$

$$= 2\beta_4\left[V_1^2+V_2^2-\underline{\underline{2}}V_{1i}V_{2i}\right]^2 + 4\beta_4(V_{2i}-V_{1i})(V_{1i}-V_{2i})(V_1^2+V_2^2-\underline{\underline{2}}V_{1k}V_{2k}) \quad + 2(V_1^2+V_2^2)\beta_2(V_1^2+V_2^2-\underline{\underline{2}}V_{1k}V_{2k})$$
$$+ 4\frac{\beta_2}{d+2}(V_{1i}V_{1j}+V_{2i}V_{2j})\left[2(V_{1i}-V_{2i})(V_{1j}-V_{2j}) + (V_1^2+V_2^2-\underline{\underline{2}}V_{1k}V_{2k})\delta_{ij}\right]$$
$$+ 4\beta_2(V_2^2V_{2i} - V_1^2V_{1i})(V_{1i}-V_{2i})$$

$$= 2\underbrace{\textcircled{\beta_4}}_{=3\beta_2/(d+2)}\left[(V_1^2+V_2^2)^2 + \underline{\underline{4}}\underbrace{V_{1i}V_{2i}V_{1j}V_{2j}}_{=(V_1\cdot V_2)^2} - \underline{\underline{4}}\underbrace{V_{1i}V_{2i}}_{=V_1\cdot V_2}(V_1^2+V_2^2)\right]$$
$$+ 4\underbrace{\textcircled{\beta_4}}_{=3\beta_2/(d+2)}\Big(\overbrace{V_{2i}V_{1i}}^{=V_1\cdot V_2} - \overbrace{V_{2i}V_{2i}}^{=V_2^2} - \underbrace{V_{1i}V_{1i}}_{=V_1^2} + \overbrace{V_{1i}V_{2i}}^{=V_1\cdot V_2}\Big)(V_1^2+V_2^2-\underline{\underline{2}}V_{1k}V_{2k})$$
$$+ 2\beta_2(V_1^2+V_2^2)(V_1^2+V_2^2-\underline{\underline{2}}V_{1k}V_{2k})$$
$$+ 4\frac{\beta_2}{d+2}(V_{1i}V_{1j}+V_{2i}V_{2j})\left[2V_{1i}V_{1j} - 2V_{1i}V_{2j} - 2V_{2i}V_{1j} + 2V_{2i}V_{2j} + (V_1^2+V_2^2-\underline{\underline{2}}V_{1k}V_{2k})\delta_{ij}\right]$$
$$+ 4\beta_2\Big(V_2^2\overbrace{V_{2i}V_{1i}}^{=V_1\cdot V_2} + V_2^2\overbrace{V_{2i}V_{2i}}^{=V_2^2} - V_1^2\overbrace{V_{1i}V_{1i}}^{=V_1^2} + V_1^2\overbrace{V_{1i}V_{2i}}^{=V_1\cdot V_2}\Big)$$

$$;\ \beta_4 = 2\pi^{(d-1)/2}\frac{\Gamma\left(\frac{1+4}{2}\right)}{\Gamma\left(\frac{d+4}{2}\right)}$$
$$= 2\pi^{(d-1)/2}\frac{\Gamma\left(\frac{1+2}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right)}\frac{\frac{3}{\cancel{2}}}{\frac{d+2}{\cancel{2}}}$$
$$= \frac{3}{d+2}\beta_2$$

$$= \frac{6\beta_2}{d+2}\left[V_1^4+V_2^4+2V_1^2V_2^2 + \underline{\underline{4}}(V_1\cdot V_2)^2 - \underline{\underline{4}}(V_1^2+V_2^2)(V_1\cdot V_2)\right]$$
$$+ \frac{12\beta_2}{d+2}\left[-V_1^2-V_2^2+2(V_1\cdot V_2)\right]\left[V_1^2+V_2^2-\underline{\underline{2}}(V_1\cdot V_2)\right]$$
$$+ 2\beta_2\left[V_1^2+V_2^2\right]\left[V_1^2+V_2^2-\underline{\underline{2}}(V_1\cdot V_2)\right]$$
$$+ \frac{4\beta_2}{d+2}\Big[2\overbrace{V_{1i}V_{1j}V_{1i}V_{1j}}^{=V_1^4} - 2\overbrace{V_{1i}V_{1j}V_{1i}V_{2j}}^{=(V_1\cdot V_2)V_1^2} - 2\overbrace{V_{1i}V_{1j}V_{2i}V_{1j}}^{=(V_1\cdot V_2)V_1^2} + 2\overbrace{V_{1i}V_{1j}V_{2i}V_{2j}}^{=(V_1\cdot V_2)^2}$$
$$+2\underbrace{V_{2i}V_{2j}V_{1i}V_{1j}}_{=(V_1\cdot V_2)^2} - 2\underbrace{V_{2i}V_{2j}V_{1i}V_{2j}}_{=(V_1\cdot V_2)V_2^2} - 2\underbrace{V_{2i}V_{2j}V_{2i}V_{1j}}_{=(V_1\cdot V_2)V_2^2} + 2\underbrace{V_{2i}V_{2j}V_{2i}V_{2j}}_{=V_2^4}\Big]$$
$$+ \frac{4\beta_2}{d+2}\left[V_1^2+V_2^2-\underline{\underline{2}}(V_1\cdot V_2)\right]\Big[\underbrace{V_{1i}V_{1j}\delta_{ij}}_{=V_1^2} + \underbrace{V_{2i}V_{2j}\delta_{ij}}_{=V_2^2}\Big]$$
$$+ 4\beta_2\left[V_2^4 - V_1^4 + (V_1\cdot V_2)(V_1^2+V_2^2)\right]$$

$$= \frac{2\beta_2}{d+2}\Big\{ 3V_1^4 + 3V_2^4 + 6V_1^2V_2^2 \boxed{+12(V_1\cdot V_2)^2} - 12(V_1\cdot V_2)(V_1^2+V_2^2)$$

$$\left[-6V_1^2 - 6V_2^2 + 12(V_1\cdot V_2)\right]\left[V_1^2+V_2^2-\cancel{2}(V_1\cdot V_2)\right]$$

$$+(d+2)(V_1^2+V_2^2)\left[V_1^2+V_2^2-\cancel{2}(V_1\cdot V_2)\right]$$

$$+4V_1^4 - 8V_1^2(V_1\cdot V_2) \boxed{+8(V_1\cdot V_2)^2} - 8V_2^2(V_1\cdot V_2) + 4V_2^4$$

$$+2(V_1^2+V_2^2)\left[V_1^2+V_2^2-\cancel{2}(V_1\cdot V_2)\right]$$

$$+2(d+2)(V_2^4-V_1^4) + 2(d+2)(V_1\cdot V_2)(V_1^2+V_2^2)\Big\}$$

$$= \frac{2\beta_2}{d+2}\Big\{ \left[V_1^2+V_2^2-\cancel{2}(V_1\cdot V_2)\right]\left[-6V_1^2-6V_2^2+12(V_1\cdot V_2) + (d+2)V_1^2 + (d+2)V_2^2 + 2V_1^2 + 2V_2^2\right]$$

$$+3V_1^4 + 3V_2^4 + \cancel{4V_1^4} + 4V_2^4 + 2dV_2^4 + 4V_2^4 - 2dV_1^4 - \cancel{4V_1^4}$$

$$-12(V_1\cdot V_2)V_1^2 - 12(V_1\cdot V_2)V_2^2 - 8V_1^2(V_1\cdot V_2) - 8V_2^2(V_1\cdot V_2) + 2(d+2)V_1^2(V_1\cdot V_2) + 2(d+2)V_2^2(V_1\cdot V_2)$$

$$+12(V_1\cdot V_2)^2 + 8(V_1\cdot V_2)^2$$

$$+6V_1^2V_2^2\Big\}$$

$$= \frac{2\beta_2}{d+2}\Big\{ \left[V_1^2+V_2^2-\cancel{2}(V_1\cdot V_2)\right]\left[(d-2)V_1^2 + (d-2)V_2^2 + 12(V_1\cdot V_2)\right]$$

$$+(3-2d)V_1^4 + (11+2d)V_2^4$$

$$+2(d-8)V_1^2(V_1\cdot V_2) + 2(d-8)V_2^2(V_1\cdot V_2)$$

$$+20(V_1\cdot V_2)^2$$

$$+6V_1^2V_2^2\Big\}$$

$$= \frac{2\beta_2}{d+2}\Big\{ (d-2)V_1^4 + (d-2)V_1^2V_2^2 + 12V_1^2(V_1\cdot V_2) + (d-2)V_1^2V_2^2 + \boxed{(d-2)V_2^4} + 12V_2^2(V_1\cdot V_2)$$

$$-2(d-2)V_1^2(V_1\cdot V_2) - 2(d-2)V_2^2(V_1\cdot V_2) - 24(V_1\cdot V_2)^2$$

$$+(3-2d)V_1^4 + \boxed{(11+2d)V_2^4} + 2(d-8)V_1^2(V_1\cdot V_2) + 2(d-8)V_2^2(V_1\cdot V_2) + 20(V_1\cdot V_2)^2 + 6V_1^2V_2^2\Big\}$$

$$= \frac{2\beta_2}{d+2}\Big[\underset{\checkmark}{-(d-1)V_1^4} + \underset{\checkmark}{3(d+3)V_2^4} + \underset{\checkmark}{(2d+2)V_1^2V_2^2} - 4(V_1\cdot V_2)^2 + V_1^2(V_1\cdot V_2)(12-2d+4+2d-16)$$

$$+ V_2^2(V_1\cdot V_2)\big(\underbrace{12-2d+4+2d-16}_{=0}\big)\Big]$$

$$= \frac{2\beta_2}{d+2}\left[-(d-1)V_1^4 + 3(d+3)V_2^4 + 2(d+1)V_1^2V_2^2 - 4(V_1\cdot V_2)^2\right]$$

↖ne reste que des termes pairs!

Ainsi:

$$\int_{\mathbb{R}^d} dV_1\, V_1^4 L_c[\mathcal{M}S_2] = -\sigma^{d-1}\frac{\varnothing}{S_d V_T}\int_{\mathbb{R}^{2d}} dV_1 dV_2 \frac{n^2}{V_T^{2d}}\frac{1}{(\pi^{d})^2} e^{-V_1^2/V_T^2} e^{-V_2^2/V_T^2} S_2(V_2^2)\Big[V_1^2+V_2^2-\underbrace{2\overset{\nearrow 0}{\cancel{(V_1\cdot V_2)}}}_{\text{impair}}\Big]\frac{2\beta_2}{d+2}\Big[\underbrace{\;\cdots\;}_{\text{pair}}\Big]$$

$$= -\sigma^{d-1}\frac{\varnothing}{S_d V_T}\frac{2\beta_2}{d+2}\frac{n^2}{V_T^{2d}}\frac{1}{\pi^d}\int_{\mathbb{R}^{2d}} dV_1 dV_2\, e^{-V_1^2/V_T^2} e^{-V_2^2/V_T^2} S_2(V_2^2)\left[V_1^2+V_2^2\right]\Big[-(d-1)V_1^4 + 3(d+3)V_2^4 + \overbrace{2(d+1)V_1^2V_2^2 - \underbrace{4(V_1\cdot V_2)^2}_{-\frac{4}{d}V_1^2V_2^2}}^{V_1^2V_2^2\frac{2d(d+1)-4}{d}}\Big]$$

$$= -\sigma^{d-1}\frac{\varnothing}{S_d V_T}\frac{2\beta_2}{d+2}\frac{n^2}{\cancel{V_T^{2d}}}\frac{1}{\pi^d}\cancel{V_T^{2d}}\, V_T^6\int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2} e^{-c_2^2} S_2(c_2^2)\left[c_1^2+c_2^2\right]\left[-(d-1)c_1^4 + 3(d+3)c_2^4 + \frac{2d(d+1)-4}{d}c_1^2c_2^2\right]$$

$$= -\sigma^{d-1}\frac{\varnothing}{S_d} V_T^5\frac{2\beta_2}{d+2}\frac{n^2}{\pi^d}\int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2} e^{-c_2^2} S_2(c_2^2)\Big[-\overset{\int dc_2 \to 0}{\cancel{(d-1)c_1^6}} + 3(d+3)c_1^2c_2^4 + \frac{2d(d+1)-4}{d}c_1^4\overset{\to 0\,:\,\text{cf. ⊛ p.16 } \int dc_2}{c_2^2}$$

$$-(d-1)c_1^4\underset{\int dc_2 \to 0}{\cancel{c_2^2}} + 3(d+3)c_2^6 + \frac{2d(d+1)-4}{d}c_1^2c_2^4\Big]$$

$$= -\sigma^{d-1}\frac{\varnothing}{S_d} V_T^5\frac{2\beta_2}{d+2}\frac{n^2}{\pi^d}\int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2} e^{-c_2^2} S_2(c_2^2)\Big[3(d+3)c_2^6 + c_1^2c_2^4\underbrace{\frac{2d(d+1)-4+3d(d+3)}{d}}_{=\frac{1}{d}\{2d^2+2d+3d^2+9d-4\} = \frac{1}{d}\{5d^2+11d-4\}}\Big]$$

$$= -\sigma^{d-1}\frac{\varnothing}{S_d} V_T^5\frac{2\beta_2}{d+2}\frac{n^2}{\pi^d}\Big[\underbrace{\int_{\mathbb{R}^d} dc_1 e^{-c_1^2}}_{=\pi^{d/2}}\underbrace{\int_{\mathbb{R}^d} dc_2 e^{-c_2^2} S_2(c_2^2)c_2^6\, 3(d+3)}_{=3(d+3)\pi^{d/2}\frac{3d(d+2)(d+4)}{8}} + \frac{5d^2+11d-4}{d}\underbrace{\int_{\mathbb{R}^d} dc_1 e^{-c_1^2}c_1^2}_{=\pi^{d/2}\frac{d}{2}}\underbrace{\int_{\mathbb{R}^d} dc_2 e^{-c_2^2}c_2^4 S_2(c_2^2)}_{=\pi^{d/2}\frac{d(d+2)}{4}}\Big]$$

$$= -\sigma^{d-1}\frac{\varnothing}{S_d} V_T^5\frac{\cancel{2}\beta_2}{\cancel{d+2}}\frac{n^2}{\cancel{\pi^d}}\cancel{\pi^d}\frac{d\cancel{(d+2)}}{4}\frac{1}{\cancel{2}}\Big[3(d+3)\,3(d+4) + \frac{5d^2+11d-4}{\cancel{d}}\cancel{d}\Big]$$

$$= -\sigma^{d-1}\varnothing V_T^5 n^2\frac{d}{4}\cancel{2}\pi^{(d-1)/2}\frac{\Gamma\left(\frac{d+2}{2}\right)}{\underbrace{\Gamma\left(\frac{d+2}{2}\right)}_{=\frac{1}{2}\sqrt{\pi}\,/\,\frac{d}{2}\Gamma(d/2)}}\frac{\Gamma(d/2)}{\cancel{2}\pi^{d/2}}\Big[\underbrace{9(d+3)(d+4) + 5d^2+11d-4}_{=14d^2+74d+105\;=}\Big]$$

Terme de collision : vérification **CORRECTION** ①

$$\int_{\mathbb{R}^d} dV_1 V_1^4 L_c[MS_2] = -\sigma^{d-1}\frac{\emptyset}{S_d V_T}\int_{\mathbb{R}^{2d}} dV_1 dV_2 \underbrace{V_{12}^2 f^{(0)}(V_1)M(V_2)S_2(V_2^2)}\int d\hat{\sigma}\,(b-1)\left[V_1^4+V_2^4\right]$$

$$= (V_1^2+V_2^2-2V_1\cdot V_2)\frac{n^2}{V_T^{2d}}\frac{1}{\pi^d}\,e^{-V_1^2/V_T^2}e^{-V_2^2/V_T^2}\left[\frac{1}{2}\frac{V_2^4}{V_T^4}-\frac{d+2}{2}\frac{V_2^2}{V_T^2}+\frac{d(d+2)}{8}\right]$$

$$= -\sigma^{d-1}\frac{\emptyset}{S_d\cancel{V_T}}\frac{n^2}{\cancel{V_T^{2d}}}\frac{1}{\pi^d}\cancel{V_T^{2d}}\,V_T^{5}\int_{\mathbb{R}^{2d}} dc_1dc_2\,(c_1^2+c_2^2-2c_1c_2)e^{-c_1^2}e^{-c_2^2}\left[\frac{1}{2}c_2^4-\frac{d+2}{2}c_2^2+\frac{d(d+2)}{8}\right]\int d\hat{\sigma}\,(b-1)\left[c_1^4+c_2^4\right]$$

$$= -\sigma^{d-1}\frac{\emptyset n^2}{S_d\pi^d}V_T^5\int_{\mathbb{R}^{2d}} dc_1dc_2\,e^{-c_1^2}e^{-c_2^2}\underbrace{\left[\frac{1}{2}c_2^4-\frac{d+2}{2}c_2^2+\frac{d(d+2)}{8}\right]}_{=S_2(c_2^2)}(c_1^2+c_2^2-2c_1\cdot c_2)\int d\hat{\sigma}\,(b-1)\left[c_1^4+c_2^4\right] \quad ; b\underline{c}_i = \underline{c}_i \mp (g\hat{\sigma})\hat{\sigma}$$

Intégrale angulaire :

$$bc_1^4 = \left[c_1^2+(g\cdot\hat{\sigma})^2-2(g\cdot\hat{\sigma})(c_1\cdot\hat{\sigma})\right]^2$$

$$= c_1^4+(g\cdot\hat{\sigma})^4+2c_1^2(g\cdot\hat{\sigma})^2+4(g\cdot\hat{\sigma})^2(c_1\cdot\hat{\sigma})^2-4(g\cdot\hat{\sigma})(c_1\cdot\hat{\sigma})c_1^2-4(g\cdot\hat{\sigma})^3(c_1\cdot\hat{\sigma})$$

$$bc_2^4 = c_2^4+(g\hat{\sigma})^4+2c_2^2(g\cdot\hat{\sigma})^2+4(g\cdot\hat{\sigma})^2(c_2\cdot\hat{\sigma})^2+4(g\cdot\hat{\sigma})(c_2\hat{\sigma})c_2^2+4(g\cdot\hat{\sigma})^3(c_2\cdot\hat{\sigma})$$

$$(b-1)\left[c_1^4+c_2^4\right] = 2(g\cdot\hat{\sigma})^4+2(g\cdot\hat{\sigma})^2(c_1^2+c_2^2)+4(g\cdot\hat{\sigma})^2(c_{1i}c_{1j}+c_{2i}c_{2j})\sigma_i\sigma_j+4(g\cdot\hat{\sigma})\left[c_2^2c_{2i}-c_1^2c_{1i}\right]\sigma_i$$
$$+4(g\cdot\hat{\sigma})^3\left[c_{2i}-c_{1i}\right]\sigma_i$$

$$= 2(g\cdot\hat{\sigma})^4+4(g\cdot\hat{\sigma})^3\sigma_i(c_{2i}-c_{1i})+2(g\cdot\hat{\sigma})^2(c_1^2+c_2^2)+4(g\cdot\hat{\sigma})^2\sigma_i\sigma_j(c_{1i}c_{1j}+c_{2i}c_{2j})$$
$$+4(g\cdot\hat{\sigma})\sigma_i(c_2^2c_{2i}-c_1^2c_{1i})$$

$$\int d\hat{\sigma}\,(b-1)\left[V_1^4+V_2^4\right] = 2\beta_4g^4+4(c_{2i}-c_{1i})\beta_4g^2g_i+2(c_1^2+c_2^2)\beta_2g^2+4(c_{1i}c_{1j}+c_{2i}c_{2j})\frac{\beta_2}{d+2}(2g_ig_j+g^2\delta_{ij})$$
$$+4(c_2^2c_{2i}-c_1^2c_{1i})\beta_2g_i$$

Or:

$$\beta_n = 2\pi^{(d-1)/2}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+d}{2}\right)} = 2\pi^{(d-1)/2}\frac{n-1}{2}\frac{\Gamma\left[\frac{(n-2)+1}{2}\right]}{\frac{n-2+d}{2}\Gamma\left[\frac{(n-2)+d}{2}\right]} = \frac{n-1}{2}\frac{2}{n-2+d}\beta_{n-2} = \frac{n-1}{n-2+d}\beta_{n-2}$$

$$\Rightarrow \beta_4 = \frac{3}{d+2}\beta_2$$

Ainsi:

$$\int d\hat{\sigma}\,(b-1)\left[V_1^4+V_2^4\right] = \frac{6}{d+2}\beta_2g^4+\frac{12}{d+2}\beta_2\overbrace{(c_{2i}-c_{1i})}^{=-g_i}g^2g_i+2\beta_2(c_1^2+c_2^2)g^2+\frac{4\beta_2}{d+2}(c_{1i}c_{1j}+c_{2i}c_{2j})(2g_ig_j+g^2\delta_{ij})$$
$$+4\beta_2(c_2^2c_{2i}-c_1^2c_{1i})g_i$$

$$= \frac{2\beta_2}{d+2}\left[3g^4-6g^4+(d+2)(c_1^2+c_2^2)g^2+2(c_{1i}c_{1j}+c_{2i}c_{2j})(2g_ig_j+g^2\delta_{ij})+2(d+2)g_i(c_2^2c_{2i}-c_1^2c_{1i})\right]$$

$$= \frac{2\beta_2}{d+2}\left[-3g^4+(d+2)g^2(c_1^2+c_2^2)+2g^2(c_1^2+c_2^2)+4g_ig_j(c_{1i}c_{1j}+c_{2i}c_{2j})+2(d+2)(c_2^2c_{2i}g_i-c_1^2c_{1i}g_i)\right]$$

$$= \frac{2\beta_2}{d+2}\Big[-3g^4+(d+4)g^2\overbrace{(c_1^2+c_2^2)}^{=c_1^2+c_2^2-2c_1c_2+2c_1c_2}+4g_ig_j(c_{1i}c_{1j}+c_{2i}c_{2j})+2(d+2)(c_2^2\overbrace{c_{2i}c_{1i}}^{c_1\cdot c_2}-c_2^2\overbrace{c_{2i}c_{2i}}^{c_2^2}$$
$$-c_1^2\underbrace{c_{1i}c_{1i}}_{c_1^2}+c_1^2\underbrace{c_{1i}c_{2i}}_{c_1\cdot c_2})\Big]$$

$$= \frac{2\beta_2}{d+2}\Big[-3g^4+(d+4)g^2(g^2+2c_1\cdot c_2)+4(c_{1i}c_{1j}-c_{1i}c_{2j}-c_{2i}c_{1j}+c_{2i}c_{2j})(c_{1i}c_{1j}+c_{2i}c_{2j})$$
$$+2(d+2)(c_1\cdot c_2)(c_1^2+c_2^2)-2(d+2)(c_1^4+c_2^4)\Big]$$

$$= \frac{2\beta_2}{d+2}\Big[(d+1)g^4+2(d+4)g^2(c_1\cdot c_2)+2(d+2)\overbrace{(c_1^2+c_2^2-2c_1\cdot c_2+2c_1\cdot c_2)}^{=g^2+2c_1\cdot c_2}c_1\cdot c_2-2(d+2)(c_1^4+c_2^4)$$
$$+4\Big(\overbrace{c_{1i}c_{1j}c_{1i}c_{1j}}^{c_1^4}+\overbrace{c_{1i}c_{1j}c_{2i}c_{2j}}^{(c_1\cdot c_2)^2}-\overbrace{c_{1i}c_{2j}c_{1i}c_{1j}}^{c_1^2(c_1\cdot c_2)}-\overbrace{c_{1i}c_{2j}c_{2i}c_{2j}}^{c_2^2(c_1\cdot c_2)}-\overbrace{c_{2i}c_{1j}c_{1i}c_{1j}}^{c_1^2(c_1\cdot c_1)}-\overbrace{c_{2i}c_{1j}c_{2i}c_{2j}}^{c_2^2(c_1\cdot c_2)}$$
$$+\overbrace{c_{2i}c_{2j}c_{1i}c_{1j}}^{=(c_1c_2)^2}+\overbrace{c_{2i}c_{2j}c_{2i}c_{2j}}^{c_2^4}\Big)\Big]$$

$$= \frac{2\beta_2}{d+2}\Big[(d+1)g^4+g^2(c_1\cdot c_2)2(d+4+d+2)+4(d+2)(c_1\cdot c_2)^2-2(d+2)(c_1^4+c_2^4)+4(c_1^4+c_2^4)+8(c_1\cdot c_2)^2$$
$$-8c_1^2(c_1\cdot c_2)-8c_2^2(c_1\cdot c_2)\Big]$$

$$= \frac{2\beta_2}{d+2}\Big[(d+1)g^4+4(d+3)g^2(c_1\cdot c_2)+4(d+4)(c_1\cdot c_2)^2-2d(c_1^4+c_2^4)-4(c_1\cdot c_2)\overbrace{(c_1^2+c_2^2)}^{=g^2+2c_1\cdot c_2}\Big]$$

$$= \frac{2\beta_2}{d+2}\Big[(d+1)g^4+\overbrace{(4d+12-4)}^{=4(d+2)}g^2(c_1\cdot c_2)+\overbrace{(4d+16-8)}^{=4(d+2)}(c_1\cdot c_2)^2-2d(c_1^4+c_2^4)\Big]$$

avec :

$$g^2 = c_1^2+c_2^2-2c_1\cdot c_2$$
$$g^4 = c_1^4+c_2^4+2c_1^2c_2^2+4(c_1\cdot c_2)^2-4(c_1\cdot c_2)(c_1^2+c_2^2)$$

alors :

②

$$\int d\hat\sigma\,(b-1)[V_1^4+V_2^4] = \frac{2\beta_2}{d+2}\Big[\,(d+1)(c_1^4+c_2^4) + 2(d+1)c_1^2c_2^2 + \underline{4(d+1)(c_1\cdot c_2)^2} - 4(d+1)(c_1\cdot c_2)(c_1^2+c_2^2)$$
$$+4(d+2)(c_1\cdot c_2)(c_1^2+c_2^2) \; \underline{+4(d+2)(c_1\cdot c_2)^2(-2)} \; \underline{+4(d+2)(c_1\cdot c_2)^2} - 2d(c_1^4+c_2^4)\,\Big]$$
$$= \frac{2\beta_2}{d+2}\Big[-(d-1)(c_1^4+c_2^4) + 2(d+1)c_1^2c_2^2 + 4(c_1\cdot c_2)(c_1^2+c_2^2) - 4(c_1\cdot c_2)^2\Big] \checkmark$$

$$\int_{\mathbb{R}^d} dv_1\, V_1^4 \mathcal{L}_c[MS_2] = -\sigma^{d-1}\frac{\phi n^2}{\sqrt{d}\,\pi^d}V_T^5\frac{2\beta_2}{d+2}\int_{\mathbb{R}^{2d}} dc_1dc_2\, e^{-c_1^2}e^{-c_2^2} S_2(c_2^2)(c_1^2+c_2^2-2c_1\cdot c_2)\Big[\overbrace{-(d-1)(c_1^4+c_2^4)+2(d+1)c_1^2c_2^2-4(c_1c_2)^2}^{pair}+\overbrace{4(c_1^2+c_2^2)(c_1\cdot c_1)}^{impair}\Big]$$

$$= -\sigma^{d-1}\frac{\phi n^2}{\pi^d}\frac{\Gamma(d/2)}{2\pi^{d/2}}V_T^5\frac{2}{d+2}2\pi^{(d-1)/2}\frac{\Gamma(3/2)}{\Gamma(\frac{d+2}{2})}\int_{\mathbb{R}^{2d}} dc_1dc_2\, e^{-c_1^2}e^{-c_2^2} S_2(c_2^2)\Big[-(d-1)(c_1^4+c_2^4)(c_1^2+c_2^2)+2(d+1)c_1^2c_2^2(c_1^2+c_2^2)$$
$$\underline{-4(c_1\cdot c_2)^2(c_1^2+c_2^2)} - 8(c_1^2+c_2^2)(c_1\cdot c_2)^2\Big]$$

$$= -\sigma^{d-1}\frac{\phi n^2}{\pi^d}\frac{\cancel{\Gamma(d/2)}}{\cancel{2\pi^{d/2}}}V_1^5\frac{\cancel{2}}{d+2}\cancel{2\pi^{d/2}}\cancel{\pi^{-1/2}}\,\cancel{\tfrac{1}{2}}\cancel{\pi^{1/2}}\frac{1}{\frac{d}{2}\cancel{\Gamma(d/2)}}\int_{\mathbb{R}^{2d}} dc_1dc_2\, e^{-c_1^2}e^{-c_2^2} S_2(c_2^2)\Big[-(d-1)\cancelto{0}{c_1^6}-(d-1)c_1^4\cancelto{0}{c_2^2}-(d-1)c_1^2c_2^4-(d-1)c_2^6$$
$$+2(d+1)c_1^2c_2^2(c_1^2+c_2^2) - \frac{12}{d}c_1^2c_2^2(c_1^2+c_2^2)\Big]$$

$$= -\sigma^{d-1}\frac{\phi n^2}{\pi^d}V_T^5\frac{2}{d(d+2)}\int_{\mathbb{R}^{2d}} dc_1dc_2\, e^{-c_1^2}e^{-c_2^2} S_2(c_2^2)\Big[-(d-1)c_1^2c_2^4-(d-1)c_2^6+\frac{2d(d+1)-12}{d}\cancelto{0}{c_1^4}c_2^2+\frac{2d(d+1)-12}{d}c_1^2c_2^4\Big]$$

$$= -\sigma^{d-1}\frac{\phi n^2}{\pi^d}V_T^5\frac{2}{d(d+2)}\int_{\mathbb{R}^{2d}} dc_1dc_2\, e^{-c_1^2}e^{-c_2^2} S_2(c_2^2)\Big[-(d-1)c_2^6 + c_1^2c_2^4\,\frac{2d(d+1)-12-d(d-1)}{d}\Big]$$

$$= 2d^2+2d-12-d^2+d$$
$$= d^2+3d-12$$

$$= -\sigma^{d-1}\frac{\phi n^2}{\pi^d}V_T^5\frac{2}{d(d+2)}\pi^d\Big[-(d-1)\frac{d+4}{2}\frac{d+2}{2}\frac{d}{2} + \frac{d^2+3d-12}{\cancel{d}}\,\frac{\cancel{d}}{2}\frac{d+2}{2}\frac{d}{2}\Big]$$

$$= \cancel{-d^2}-4d+d+4+\cancel{d^2}+3d-12 = -8$$

$$= -\sigma^{d-1}\frac{\phi n^2}{\pi^d}V_T^5\frac{2}{\cancel{d(d+2)}}\frac{\cancel{d(d+2)}}{8}\Big[-(d-1)(d+4)+d^2+3d-12\Big]$$

$$= 2\sigma^{d-1}\phi n^2 V_T^5$$

$$= -\sigma^{d-1}\varnothing V_T^5 n^2 \frac{\cancel{1}}{4} \cancel{\pi^{1/2}} \frac{1}{\cancel{4}} \cancel{\pi^{1/2}} \frac{\cancel{1}}{\cancel{4}} \frac{\cancel{\Gamma(d/2)}}{\cancel{\Gamma(d/2)}} (14d^2+74d+105)$$

$$= \cancel{-\sigma^{d-1}\varnothing V_T^5 \frac{n^2}{4}(14d^2+74d+105)}$$ ⟵ je pense qu'il y a une erreur ici : doit être positif et plus grand.

$2\sigma^{d-1}\varnothing n^2 V_T^5$

Conclusion :

$$\boxed{\mathcal{V}_\xi^* = \frac{m^2\beta^2}{d(d+2)V_0 n}\sigma^{d-1}\varnothing V_T^5 n^2\left[p\frac{d(d+2)(d+3)}{2} + (1-p)2\right]}$$

3) Equations pour $C_n^{(2)}$ et $C_T^{(2)}$

$$\begin{cases}\left(2p\xi_n^{(0)*} - p\frac{5}{2}\xi_T^{(0)*} + \mathcal{V}_\xi^*\right)C_T^{(2)} = p\xi_n^{(0)*}\frac{n}{2T}C_n^{(2)} + p\overset{0}{\cancel{\xi_{n,1}^{(2)}}}\frac{\overset{=0}{\cancel{a_2}}+1}{2}\frac{m\beta}{V_0} + p\frac{\xi_{T,1}^{(2)}}{V_0}2(\overset{=0}{\cancel{a_2}}+1) + \frac{8K\beta}{d(d+2)nV_0}\left[1-\frac{d+2}{2}\overset{=0}{\cancel{a_2}}\right]\\ \left(p\xi_n^{(0)*} + \mathcal{V}_\xi^*\right)C_n^{(2)} = p\xi_T^{(0)*}\frac{T}{n}C_T^{(2)} + p\overset{0}{\cancel{\xi_{n,2}^{(2)}}}\frac{\overset{=0}{\cancel{a_2}}+1}{2}\frac{m\beta}{V_0} + p\frac{\xi_{T,2}^{(2)}}{V_0}2(\overset{=0}{\cancel{a_2}}+1) + \frac{8M\beta}{d(d+2)nV_0}\left[1-\frac{d+2}{2}\overset{=0}{\cancel{a_2}}\right]\end{cases}$$

On adimensionalise :

$$\begin{cases}\left(2p\xi_n^{(0)*} - p\frac{5}{2}\xi_T^{(0)*} + \mathcal{V}_\xi^*\right)C_T^{(2)} = p\xi_n^{(0)*}\frac{n}{2T}C_n^{(2)} + p\,2\frac{1}{V_0}\underbrace{\xi_{T,1}^{(2)}\frac{d\,nk_BT}{(d-1)K_0}}_{=\xi_{T,1}^{(2)*}}\frac{(d-1)K_0}{d\,nk_BT} + \frac{8\beta}{d(d+2)nV_0}\overset{=K^*}{\frac{K}{K_0}}K_0 & \Big|\cdot\frac{nV_0}{K_0\beta}\\ \left(p\xi_n^{(0)*} + \mathcal{V}_\xi^*\right)C_n^{(2)} = p\xi_T^{(0)*}\frac{T}{n}C_T^{(2)} + p\frac{2}{V_0}\underbrace{\xi_{T,2}^{(2)}\frac{dn^2k_B}{(d-1)K_0}}_{=\xi_{T,2}^{(2)*}}\frac{(d-1)K_0}{dn^2k_B} + \frac{8\beta}{d(d+2)nV_0}\overset{=M^*}{\frac{Mn}{TK_0}}\frac{TK_0}{n} & \Big|\cdot\frac{n^2V_0}{TK_0\beta}\end{cases}$$

$$\Rightarrow\begin{cases}\left(2p\xi_n^{(0)*} - p\frac{5}{2}\xi_T^{(0)*} + \mathcal{V}_\xi^*\right)C_T^{(2)}\frac{nV_0}{K_0\beta} = p\xi_n^{(0)*}C_n^{(2)}\frac{n^2V_0}{2TK_0\beta} + p\frac{2}{\cancel{V_0}}\frac{(d-1)\cancel{K_0}}{d\cancel{n}\cancel{k_BT\beta}}\frac{\cancel{n}\cancel{V_0}}{\cancel{K_0}}\xi_{T,1}^{(2)*} + \frac{8\cancel{\beta}\cancel{K_0}}{d(d+2)\cancel{nV_0}}\frac{\cancel{nV_0}}{\cancel{K_0\beta}}K^*\\ \left(p\xi_n^{(0)*} + \mathcal{V}_\xi^*\right)C_n^{(2)}\frac{n^2V_0}{TK_0\beta} = p\xi_T^{(0)*}\frac{\cancel{T}}{\cancel{n}}C_T^{(2)}\frac{n^{\cancel{2}}V_0}{\cancel{T}K_0\beta} + p\frac{2}{\cancel{V_0}}\frac{(d-1)\cancel{K_0}}{d\cancel{n^2}k_B\cancel{TV_0\beta}}\frac{\cancel{n^2V_0}}{}\xi_{T,2}^{(2)*} + \frac{8\cancel{\beta}}{d(d+2)\cancel{nV_0}}\frac{\cancel{TK_0}}{\cancel{n}}\frac{\cancel{n^2V_0}}{\cancel{TK_0\beta}}M^*\end{cases}$$

$$\Rightarrow\begin{cases}\left(2p\xi_n^{(0)*} - p\frac{5}{2}\xi_T^{(0)*} + \mathcal{V}_\xi^*\right)C_T^{(2)}\frac{nk_BTV_0}{K_0} = p\frac{1}{2}\xi_n^{(0)*}C_n^{(2)}\frac{n^2k_BV_0}{K_0} + p\frac{2(d-1)}{d}\xi_{T,1}^{(2)*} + \frac{8}{d(d+2)}K^*\\ \left(p\xi_n^{(0)*} + \mathcal{V}_\xi^*\right)C_n^{(2)}\frac{n^2k_BV_0}{K_0} = p\,\xi_T^{(0)*}C_T^{(2)}\frac{nk_BTV_0}{K_0} + p\frac{2(d-1)}{d}\xi_{T,2}^{(2)*} + \frac{8}{d(d+2)}M^*\end{cases}$$

Adimensionalisation :

$$C_T^{(2)*} = C_T^{(2)}\frac{nk_BTV_0}{K_0} \quad ; \quad C_T^{(2)} \propto n^{-2}T^{-1}$$

$$C_n^{(2)*} = C_n^{(2)}\frac{n^2k_BV_0}{K_0} \quad ; \quad C_n^{(2)} \propto n^{-3}T^{0}$$

$$\Rightarrow\begin{cases}\left(2p\xi_n^{(0)*} - p\frac{5}{2}\xi_T^{(0)*} + \mathcal{V}_\xi^*\right)C_T^{(2)*} = p\frac{1}{2}\xi_n^{(0)*}C_n^{(2)*} + p\frac{2(d-1)}{d}\xi_{T,1}^{(2)*} + \frac{8}{d(d+2)}K^*\\ \left(p\xi_n^{(0)*} + \mathcal{V}_\xi^*\right)C_n^{(2)*} = p\,\xi_T^{(0)*}C_T^{(2)*} + p\frac{2(d-1)}{d}\xi_{T,2}^{(2)*} + \frac{8}{d(d+2)}M^*\end{cases}$$

Résolution :

$$\begin{cases}AC_T^{(2)*} = BC_n^{(2)*} + C\\ DC_n^{(2)*} = EC_T^{(2)*} + F\end{cases} \Rightarrow C_n^{(2)*} = \frac{E}{D}C_T^{(2)*} + \frac{F}{D}$$

$$\Rightarrow AC_T^{(2)*} = B\frac{E}{D}C_T^{(2)*} + \frac{BF}{D} + C \Rightarrow \left(A - \frac{BE}{D}\right)C_T^{(2)*} = \frac{BF}{D} + C \Rightarrow C_T^{(2)*} = \frac{\cancel{D}}{AD-BE}\frac{BF}{\cancel{D}} + \frac{D}{AD-BE}C$$

$$C_T^{(2)*} = \frac{BF}{AD-BE} + \frac{CD}{AD-BE} = \frac{BF+CD}{AD-BE}$$

$$C_n^{(2)*} = \frac{E}{D}\frac{BF+CD}{AD-BE} + \frac{F}{D} = \frac{\cancel{EBF} + EC\cancel{D} + FA\cancel{D} - \cancel{FBE}}{\cancel{D}(AD-BE)} = \frac{EC+FA}{AD-BE}$$

$$\boxed{\begin{aligned}C_T^{(2)*} &= \frac{BF+CD}{AD-BE}\\ C_n^{(2)*} &= \frac{EC+AF}{AD-BE}\end{aligned}}$$

On doit y insérer les taux de déclin: ces derniers peuvent se réécrire sous la forme:

$$\xi_{T,1}^{(2)} = C_T^{(2)} \frac{m}{k_B T d} \sigma^{d-1} n v_T^3 \varnothing \frac{d(d+2)}{4}$$

$$= C_T^{(2)} \underbrace{\frac{8}{d+2}\frac{\pi^{(d-1)/2}}{\Gamma(d/2)} n\sigma^{d-1}\sqrt{\frac{k_BT}{m}}}_{=\nu_0} \frac{d+2}{8}\frac{\Gamma(d/2)}{\pi^{(d-1)/2}} \frac{1}{\cancel{n\sigma^{d-1}}} \sqrt{\frac{m}{k_BT}} \frac{m}{k_BT}\frac{1}{d}\cancel{\sigma^{d-1}}\cancel{n} v_T^3 \varnothing \frac{d(d+2)}{4}$$

$$= C_T^{(2)} \nu_0 \frac{d+2}{8} \frac{\cancel{d}(d+2)}{4}\frac{1}{\cancel{d}} \frac{\Gamma(d/2)}{\pi^{(d-1)/2}} \cancel{\left(\frac{m}{k_BT}\right)^{3/2}} \underbrace{2^{3/2}}_{=2\,2^{1/2}} \cancel{\left(\frac{k_BT}{m}\right)^{3/2}} \varnothing$$

$$= C_T^{(2)}\nu_0 \frac{(d+2)^2}{4}\varnothing\frac{\sqrt{2}\,\Gamma(d/2)}{4\pi^{(d-1)/2}}$$

$$\Rightarrow \frac{(d-1)\cancel{\nu_0}}{d\cancel{nk_BT}} \xi_{T,1}^{(2)*} = C_T^{(2)*}\frac{\cancel{\nu_0}}{\cancel{nk_BT}\cancel{\nu_0}} \cancel{\nu_0}\frac{(d+2)^2}{4}\varnothing\frac{\sqrt{2}\,\Gamma(d/2)}{4\pi^{(d-1)/2}}$$

$$\Rightarrow \xi_{T,1}^{(2)*} = C_T^{(2)*}\frac{d(d+2)^2}{4(d-1)}\varnothing\frac{\sqrt{2}\,\Gamma(d/2)}{4\pi^{(d-1)/2}} \qquad ; \ \varnothing = \frac{4\pi^{(d-1)/2}}{\sqrt{2}\,\Gamma(d/2)}\frac{4}{(d+2)(d+4)}$$

Et aussi:

$$\xi_{T,2}^{(2)} = C_n^{(2)}\frac{m}{k_BTd}\sigma^{d-1}\varnothing n v_T^3\frac{d(d+2)}{4}$$

$$= C_n^{(2)}\nu_0\frac{(d+2)^2}{4}\varnothing\frac{\sqrt{2}\,\Gamma(d/2)}{4\pi^{(d-1)/2}}$$

$$\Rightarrow \frac{(d-1)\cancel{\nu_0}}{d\cancel{n^2}\cancel{k_B}} \xi_{T,2}^{(2)*} = C_n^{(2)*}\frac{\cancel{\nu_0}}{\cancel{n^2}\cancel{k_B\nu_0}}\cancel{\nu_0}\frac{(d+2)^2}{4}\varnothing\frac{\sqrt{2}\,\Gamma(d/2)}{4\pi^{(d-1)/2}}$$

$$\Rightarrow \xi_{T,2}^{(2)*} = C_n^{(2)*}\frac{d(d+2)^2}{4(d-1)}\varnothing\frac{\sqrt{2}\,\Gamma(d/2)}{4\pi^{(d-1)/2}}$$

Conclusion:

$$\boxed{\begin{aligned}\xi_{T,1}^{(2)*} &= C_T^{(2)*}\frac{d(d+2)^2}{4(d-1)}\varnothing\frac{\sqrt{2}\,\Gamma(d/2)}{4\pi^{(d-1)/2}}\\ \xi_{T,2}^{(2)*} &= C_n^{(2)*}\frac{d(d+2)^2}{4(d-1)}\varnothing\frac{\sqrt{2}\,\Gamma(d/2)}{4\pi^{(d-1)/2}}\end{aligned}}$$

Dans les équations:

$$\left(2p\,\xi_n^{(0)*} - p\frac{5}{2}\xi_T^{(0)*} + \nu_\xi^*\right)C_T^{(2)*} = p\frac{1}{2}\xi_n^{(0)*}C_n^{(2)*} + p\frac{\cancel{2(d-1)}}{\cancel{d}}C_T^{(2)*}\frac{\cancel{d}(d+2)^2}{\underset{2}{\cancel{4(d-1)}}}\varnothing\frac{\sqrt{2}\,\Gamma(d/2)}{4\pi^{(d-1)/2}} + \frac{8}{d(d+2)}\kappa^*$$

$$\left(p\,\xi_n^{(0)*} + \nu_\xi^*\right)C_n^{(2)*} = p\,\xi_T^{(0)*}C_T^{(2)*} + p\frac{\cancel{2(d-1)}}{\cancel{d}}C_n^{(2)*}\frac{\cancel{d}(d+2)^2}{\underset{2}{\cancel{4(d-1)}}}\varnothing\frac{\sqrt{2}\,\Gamma(d/2)}{4\pi^{(d-1)/2}} + \frac{8}{d(d+2)}\mu^*$$

Valeurs des autres paramètres:

$$\xi_n^{(0)*} = \frac{d(d+2)}{2}\varnothing\frac{\Gamma(d/2)\sqrt{2}}{4\pi^{(d-1)/2}}$$

$$\xi_T^{(0)*} = \frac{d+2}{2}\varnothing\frac{\Gamma(d/2)\sqrt{2}}{4\pi^{(d-1)/2}}$$

$$\nu_\xi^* = \frac{\cancel{m^2\beta^2}}{d(d+2)\cancel{n}}\frac{d+2}{8}\frac{\Gamma(d/2)}{\pi^{(d-1)/2}}\frac{1}{\cancel{n\sigma^{d-1}}}\cancel{\sqrt{\frac{m}{k_BT}}}\,\cancel{\sigma^{d-1}}\,\varnothing\,\overset{\cancel{=\frac{4}{\beta^2m^2}\,2^{1/2}}}{v_T^5}\,\cancel{n^2}\left[p\frac{d(d+2)(d+3)}{2} + (1-p)2\,\cancel{\frac{14d^2+74d+105}{4}}\right]$$

$$= \frac{1}{d\cancel{(d+2)}}\frac{\cancel{(d+2)}}{8}\frac{\Gamma(d/2)}{\pi^{(d-1)/2}}\varnothing\, 4\,\cancel{2}^{1/2}\left[\dots\right]$$

$$= \frac{1}{d}\frac{\sqrt{2}\,\Gamma(d/2)}{2\pi^{(d-1)/2}}\varnothing\left[\dots\right]\frac{1}{2}\,2$$

$$= \frac{2}{d}\varnothing\frac{\sqrt{2}\,\Gamma(d/2)}{4\pi^{(d-1)/2}}\left[p\frac{d(d+2)(d+3)}{2} + (1-p)2\right] \qquad = \varnothing\frac{\sqrt{2}\,\Gamma(\frac{d}{2})}{4\pi^{(d-1)/2}}\left[p(d+2)(d+3) + (1-p)\frac{4}{d}\right]$$

κ^*, μ^* : cf. Eqs. (50) mon article

Résolution :

$$\left(2p\zeta_n^{(0)*}-p\tfrac{5}{2}\zeta_T^{(0)*}+V_\zeta^*\right)\underset{\sim}{C_T^{(2)*}} = p\tfrac{1}{2}\zeta_n^{(0)*}\underline{\underline{C_n^{(2)*}}}+p\frac{(d+2)^2}{2}\underset{\sim}{C_T^{(2)*}}\,\phi\,\frac{\sqrt{2}\,\Gamma(d/2)}{4\pi^{(d-1)/2}}+\frac{8}{d(d+2)}\kappa^*$$

$$\left(p\zeta_n^{(0)*}+V_\zeta^*\right)\underline{\underline{C_n^{(2)*}}} = p\zeta_T^{(0)*}\underset{\sim}{C_T^{(2)*}}+p\frac{(d+2)^2}{2}\underline{\underline{C_n^{(2)*}}}\,\phi\,\frac{\sqrt{2}\,\Gamma(d/2)}{4\pi^{(d-1)/2}}+\frac{8}{d(d+2)}\mu^*$$

Choix de ϕ : $\phi=\frac{4\pi^{(d-1)/2}}{\sqrt{2}\,\Gamma(d/2)}\frac{4}{(d+2)(d+4)}$ $\Longrightarrow$ (résultats dépendent de ϕ à travers κ^* et μ^*)

$$\Big(\overbrace{2p\zeta_n^{(0)*}-p\tfrac{5}{2}\zeta_T^{(0)*}+V_\zeta^*-p\frac{2(d+2)}{(d+4)}}^{A}\Big)C_T^{(2)*} = \overbrace{p\tfrac{1}{2}\zeta_n^{(0)*}}^{B}C_n^{(2)*}+\overbrace{\frac{8}{d(d+2)}\kappa^*}^{C}$$

$$\Big(\underbrace{p\zeta_n^{(0)*}+V_\zeta^*-p\frac{2(d+2)}{(d+4)}}_{D}\Big)C_n^{(2)*} = \underbrace{p\zeta_T^{(0)*}}_{E}C_T^{(2)*}+\underbrace{\frac{8}{d(d+2)}\mu^*}_{F}$$

$$\Longrightarrow\ AC_T^{(2)*}=B\left[\frac{E}{D}C_T^{(2)*}+\frac{F}{D}\right]+C=\frac{BE}{D}C_T^{(2)*}+\frac{BF}{D}+C$$

$$\Longrightarrow\ C_T^{(2)*}=\frac{D}{AD-BE}\left[\frac{BF}{D}+C\right]=\frac{BF+CD}{AD-BE}$$

et:

$$DC_n^{(2)*}=E\frac{BF+CD}{AD-BE}+F=\frac{\cancel{EBF}+ECD+FAD-\cancel{EBF}}{AD-BE}$$

$$\Longrightarrow\ C_n^{(2)*}=\frac{EC+FA}{AD-BE}$$

4) Coefficients de transport :

$$\boxed{\begin{aligned}\zeta_{T,1}^{(2)*}&=C_T^{(2)*}\frac{d(d+2)}{(d-1)(d+4)}\\ \zeta_{T,2}^{(2)*}&=C_n^{(2)*}\frac{d(d+2)}{(d-1)(d+4)}\end{aligned}}$$

$C_T^{(2)*}=\frac{BF+CD}{AD-BE}$; $A=2p\zeta_n^{(0)*}-p\frac{5}{2}\zeta_T^{(0)*}+V_\zeta^*-p\frac{2(d+2)}{(d+4)}$; $B=p\frac{1}{2}\zeta_n^{(0)*}$; $C=\frac{8\kappa^*}{d(d+2)}$

$C_n^{(2)*}=\frac{EC+FA}{AD-BE}$; $D=p\zeta_n^{(0)*}+V_\zeta^*-p\frac{2(d+2)}{(d+4)}$; $E=p\zeta_T^{(0)*}$; $F=\frac{8\mu^*}{d(d+2)}$

$$\zeta_n^{(0)*}=\frac{2d}{d+4}$$

$$\zeta_T^{(0)*}=\frac{2}{d+4}$$

$$V_\zeta^*=\frac{8}{d(d+2)(d+4)}\left[p\frac{d(d+2)(d+3)}{2}+(1-p)2\right]=p\frac{d+3}{d+4}4+(1-p)\frac{16}{d(d+2)(d+4)}$$

$$\left[\begin{aligned}\kappa^*&=\frac{d-1}{d}\,\frac{2V_\mu^*-2p\zeta_n^{(0)*}-3p\zeta_T^{(0)*}}{\mathcal{X}}\\ \mu^*&=2p\frac{d-1}{d}\frac{\zeta_T^{(0)*}}{\mathcal{X}}\\ \mathcal{X}&=V_\kappa^*\left[2V_\mu^*-2p\zeta_n^{(0)*}-3p\zeta_T^{(0)*}\right]+p\zeta_T^{(0)*}\left\{-4V_\mu^*+3p\left[\zeta_n^{(0)*}+2\zeta_T^{(0)*}\right]\right\}\\ V_\zeta^*&=p\frac{2(d+2)}{(d+4)}+(1-p)\\ V_\kappa^*=V_\mu^*&=p\frac{2(d+3)}{d+4}+(1-p)\,\cancel{\frac{+24+20d+8d^2-d^3}{2d(d+2)(d+4)}}\ \frac{4\left[(d-1)(d+4)\,\text{[illegible]}\right]}{d(d+2)(d+4)}\end{aligned}\right.$$

$\Longrightarrow$ PLOT : Mathematica !

Résolution sans fixer de valeur de $\varnothing$:

On divise les Equ. par $\varnothing \frac{\Gamma(d/2)\sqrt{2}}{4\pi^{(d-1)/2}} \frac{(d+2)(d+4)}{4}$ $\Longrightarrow$

$$\left[2p\xi_n^{(0)*} - p\frac{5}{2}\xi_T^{(0)*} + \nu_\xi^*\right] C_T^{(2)*} = p\frac{1}{2}\xi_n^{(0)*} C_n^{(2)*} + p\, C_T^{(2)*} \frac{(d+2)^{\cancel{2}}}{\cancel{2}} \cancel{\varnothing} \frac{\cancel{\sqrt{2}\Gamma(d/2)}}{\cancel{4\pi^{(d-1)/2}}} \frac{1}{\cancel{\varnothing}} \frac{\cancel{4\pi^{(d-1)/2}}}{\cancel{\Gamma(d/2)\sqrt{2}}} \frac{\cancel{4}\,2}{\cancel{(d+2)}(d+4)} + \frac{8}{d(d+2)}\kappa^* \frac{1}{\varnothing} \frac{4\pi^{(d-1)/2}}{\Gamma(d/2)\sqrt{2}} \frac{4}{(d+2)(d+4)} := Y$$

$$\left[p\xi_n^{(0)*} + \nu_\xi^*\right] C_n^{(2)*} = p\xi_T^{(0)*} C_T^{(2)*} + p\, C_n^{(2)*} \frac{(d+2)^{\cancel{2}}}{\cancel{2}} \cancel{\varnothing} \frac{\cancel{\sqrt{2}\Gamma(d/2)}}{\cancel{4\pi^{(d-1)/2}}} \frac{1}{\cancel{\varnothing}} \frac{\cancel{4\pi^{(d-1)/2}}}{\cancel{\Gamma(d/2)\sqrt{2}}} \frac{\cancel{4}\,2}{\cancel{(d+2)}(d+4)} + \frac{8}{d(d+2)}\mu^* \frac{1}{\varnothing} \frac{4\pi^{(d-1)/2}}{\Gamma(d/2)\sqrt{2}} \frac{4}{(d+2)(d+4)} = Y$$

$\Longrightarrow$

$$\boxed{\begin{aligned} \left[2p\xi_n^{(0)*} - p\frac{5}{2}\xi_T^{(0)*} + \nu_\xi^* - p\frac{2(d+2)}{d+4}\right] C_T^{(2)*} &= p\frac{1}{2}\xi_n^{(0)*} C_n^{(2)*} + \frac{8}{d(d+2)}\kappa^* Y \\ \left[p\xi_n^{(0)*} + \nu_\xi^* - p\frac{2(d+2)}{d+4}\right] C_n^{(2)*} &= p\xi_T^{(0)*} C_T^{(2)*} + \frac{8}{d(d+2)}\mu^* Y \end{aligned}}$$

$$\boxed{\begin{aligned} &\xi_n^{(0)*} = \frac{2d}{d+4} \;;\; \xi_T^{(0)*} = \frac{2}{d+4} \\ &\nu_\xi^* = p\frac{4(d+3)}{d+4} + (1-p)\frac{16}{d(d+2)(d+4)} \end{aligned}}$$

Résolution pour $C_T^{(2)*}$ et $C_n^{(2)*}$: $AC_T^{(2)*} = BC_n^{(2)*} + CY$; $DC_n^{(2)*} = E\,C_T^{(2)*} + FY$:

$$C_T^{(2)*} = \frac{BF+CD}{AD-BE} Y$$

$$C_n^{(2)*} = \frac{EC+FA}{AD-BE} Y$$

Coefficients de transport:

$$\boxed{\begin{aligned} \xi_{T,1}^{(2)*} &= C_T^{(2)*} \frac{d(d+2)^2}{4(d-1)} \varnothing \frac{\sqrt{2}\Gamma(d/2)}{4\pi^{(d-1)/2}} = \frac{BF+CD}{AD-BE} \frac{d(d+2)^{\cancel{2}}}{\cancel{4}(d-1)} \cancel{\varnothing} \frac{\cancel{\sqrt{2}\Gamma(d/2)}}{\cancel{4\pi^{(d-1)/2}}} \frac{1}{\cancel{\varnothing}} \frac{\cancel{4\pi^{(d-1)/2}}}{\cancel{\Gamma(d/2)\sqrt{2}}} \frac{\cancel{4}}{\cancel{(d+2)}(d+4)} = \frac{BF+CD}{AD-BE} \frac{d(d+2)}{(d-1)(d+4)} \\ \xi_{T,2}^{(2)*} &= C_n^{(2)*} \frac{d(d+2)^2}{4(d-1)} \varnothing \frac{\sqrt{2}\Gamma(d/2)}{4\pi^{(d-1)/2}} = \frac{EC+FA}{AD-BE} \frac{d(d+2)^{\cancel{2}}}{\cancel{4}(d-1)} \cancel{\varnothing} \frac{\cancel{\sqrt{2}\Gamma(d/2)}}{\cancel{4\pi^{(d-1)/2}}} \frac{1}{\cancel{\varnothing}} \frac{\cancel{4\pi^{(d-1)/2}}}{\cancel{\Gamma(d/2)\sqrt{2}}} \frac{\cancel{4}}{\cancel{(d+2)}(d+4)} = \frac{EC+FA}{AD-BE} \frac{d(d+2)}{(d-1)(d+4)} \end{aligned}}$$

Conclusion: les taux de déclin ne dépendent de $\varnothing$ que par les coefficients de transport. Par contre, un rapport $\kappa^*/\xi_{T,1}^{(2)*}$ ne dépend pas de la fréquence $\varnothing$! En effet, $\kappa^* \sim \varnothing^{-1}$, $\mu^* \sim \varnothing^{-1}$, $C_T^{(2)*} \sim \kappa^*$; $C_n^{(2)*} \sim \mu^*$.

Exemple:

$$\left\{\begin{aligned} \frac{\kappa^*}{\xi_{T,1}^{(2)*}} &= \kappa^* \frac{AD-BE}{B\frac{8}{d(d+2)}\mu^* + D\frac{8}{d(d+2)}\kappa^*} \frac{(d-1)(d+4)}{d(d+2)} = \frac{AD-BE}{B\frac{\mu^*}{\kappa^*}+D} \frac{\cancel{d(d+2)}}{8} \frac{(d-1)(d+4)}{\cancel{d(d+2)}} = \frac{AD-BE}{B\frac{\mu^*}{\kappa^*}+D} \frac{(d-1)(d+4)}{8} \\ \frac{\mu^*}{\kappa^*} &= 2p \frac{\cancel{d-1}}{\cancel{d}} \frac{\xi_T^{(0)*}}{\cancel{2}} \frac{\cancel{d}}{\cancel{d-1}} \frac{\cancel{2}}{2\nu_\mu^* - 2p\xi_n^{(0)*} - 3p\xi_T^{(0)*}} = \frac{2p\xi_T^{(0)*}}{2\nu_\mu^* - 2p\xi_n^{(0)*} - 3p\xi_T^{(0)*}} \\ \frac{\mu^*}{\xi_{T,2}^{(2)*}} &= \mu^* \frac{AD-BE}{E\kappa^* + A\mu^*} \frac{\cancel{d(d+2)}}{8} \frac{(d-1)(d+4)}{\cancel{d(d+2)}} = \frac{AD-BE}{E\frac{\kappa^*}{\mu^*}+A} \frac{(d-1)(d+4)}{8} \end{aligned}\right.$$

Equations hydrodynamiques à l'ordre 2 :

On a à présent une modification :

$$\partial_t T + u_i \nabla_i T + \frac{2}{n k_B d}\left(P_{ij} \nabla_j u_i + \nabla_i q_i \right) = -p T \left[\zeta_T^{(0)} + \overset{0}{\cancel{\zeta_T^{(1)}}} + \zeta_T^{(2)} \right]$$

$$\zeta_T^{(2)} = \zeta_{T,1}^{(2)} \nabla^2 T + \zeta_{T,2}^{(2)} \nabla^2 n$$

Il faut linéariser et adimensionaliser. On se concentre sur le nouveau terme :

$$-p T \zeta_T^{(2)} = -p(\delta T + T_H)\zeta_T^{(2)}$$

$$= -p\delta T\, \zeta_T^{(2)} - p T_H \zeta_T^{(2)}$$

$$= -p\,\delta T\, V_{0H}\, \zeta_T^{(2)*} - p T_H \zeta_T^{(2)}$$

Multiplie par $2/V_{0H}$ pour avoir l'équivalent de (633) :

$$= -p\,\delta T\, 2\, \zeta_T^{(2)*} - p \frac{2}{V_{0H}} T_H \zeta_T^{(2)}$$

Intègre $\int d\underline{\ell}\, e^{-i\underline{k}\cdot\underline{\ell}}$ pour passer en variables de Fourier :

$$= -2p\zeta_T^{(2)*} \underbrace{\int d\underline{\ell}\, e^{-i\underline{k}\cdot\underline{\ell}}\, \delta T}_{=\delta T_{\underline{k}}(\tau)} - 2p \frac{T_H}{V_{0H}} \int d\underline{\ell}\, e^{-i\underline{k}\cdot\underline{\ell}}\, \zeta_T^{(2)}$$

$$= -2p\zeta_T^{(2)*}\delta T_{\underline{k}}(\tau) - 2p\frac{T_H}{V_{0H}} \int d\underline{\ell}\, e^{-i\underline{k}\cdot\underline{\ell}}\, \zeta_T^{(2)}$$

Multiplie par $1/T_H$ pour avoir (634) :

$$= -2p\zeta_T^{(2)*}\Theta_{\underline{k}}(\tau) - \frac{2p}{V_{0H}} \int d\underline{\ell}\, e^{-i\underline{k}\cdot\underline{\ell}}\, \zeta_T^{(2)}$$

$$= -2p\zeta_T^{(2)*}\Theta_k(\tau) - \frac{2p}{V_{0H}} \int d\underline{\ell}\, e^{-i\underline{k}\cdot\underline{\ell}}\, \zeta_{T,1}^{(2)} \underbrace{\nabla_i\nabla_i T}_{=\left(\frac{1}{2}V_{0H}\right)^2 \frac{m}{k_B T_H} \nabla_{\ell i}\nabla_{\ell i} \underbrace{\boxed{T}}_{=T_H+\delta T}} - \frac{2p}{V_{0H}} \int d\underline{\ell}\, e^{-i\underline{k}\cdot\underline{\ell}}\, \zeta_{T,2}^{(2)} \underbrace{\nabla_i\nabla_i n}_{=\left(\frac{1}{2}V_{0H}\right)^2 \frac{m}{k_B T_H}\nabla_{\ell i}\nabla_{\ell i} \underbrace{\boxed{n}}_{=n_H+\delta n}}$$

$$= -2p\zeta_T^{(2)*}\Theta_k(\tau) - \frac{\cancel{2}p}{\cancel{V_{0H}}}\left(\frac{1}{2}V_{0H}\right)^{\cancel{2}} \frac{m}{k_B T_H} \int d\underline{\ell}\, e^{-i\underline{k}\cdot\underline{\ell}} \underbrace{\zeta_{T,1}^{(2)}}_{\text{donc : homogène}} \underbrace{\nabla_{\ell i}\nabla_{\ell i}(\overset{0}{\cancel{T_H}} + \delta T)}_{=O(\delta)} - \frac{p}{2}V_{0H}\frac{m}{k_B T_H} \int d\underline{\ell}\, e^{-i\underline{k}\cdot\underline{\ell}}\, \zeta_{T,2}^{(2)}\, \nabla_{\ell i}\nabla_{\ell i}(\overset{0}{\cancel{n_H}} + \delta n)$$

$$= -2p\zeta_T^{(2)*}\Theta_k(\tau) - \frac{p}{2}V_{0H}\frac{m}{k_B T_H}\zeta_{T,1}^{(2)} \underbrace{\int d\underline{\ell}\, e^{-i\underline{k}\cdot\underline{\ell}}\, \nabla_{\ell i}\nabla_{\ell i}\delta T}_{=-k^2\delta T_{\underline{k}}(\tau)} - \frac{p}{2}\frac{V_{0H}m}{k_B T_H}\Bigg(\underbrace{\int d\underline{\ell}\, e^{-i\underline{k}\cdot\underline{\ell}}\, \nabla_{\ell i}\nabla_{\ell i}\delta n}_{=-k^2\delta n_{\underline{k}}(\tau)}\Bigg)\zeta_{T,2}^{(2)}$$

$$= -2p\zeta_T^{(2)*}\Theta_k(\tau) - \frac{p}{2}\frac{V_{0H}m}{k_B \boxed{T_H}}\zeta_{T,1}^{(2)}(-k^2)\underbrace{\delta T_{\underline{k}}(\tau)}_{=\Theta_k(\tau)} - \frac{p}{2}\frac{V_{0H}m}{k_B T_H}(-k^2)\zeta_{T,2}^{(2)}\, n_H \underbrace{\frac{\delta n_{\underline{k}}(\tau)}{n_H}}_{=\rho_{\underline{k}}(\tau)}$$

$$= -2p\zeta_T^{(2)*}\Theta_{\underline{k}}(\tau) + \frac{p}{2}K^2\zeta_{T,1}^{(2)}\frac{V_{0H}m}{k_B}\Theta_k(\tau) + \frac{p}{2}K^2\zeta_{T,2}^{(2)}\frac{V_{0H}m\, n_H}{k_B T_H}\rho_{\underline{k}}(\tau) \quad ; \; K_0 = \frac{d(d+2)}{2(d-1)}\frac{k_B}{m}\eta_0 \; ; \; \eta_0 = \frac{n k_B T}{V_0}$$

Adimensionalisation des coefficients :

$$K_0 = \frac{d(d+2)}{2(d-1)}\frac{k_B}{m}\frac{n k_B T}{V_0} \Rightarrow \begin{cases} \dfrac{V_0 m}{k_B} = \dfrac{2(d-1)}{d(d+2)}\dfrac{n k_B T}{K_0} \\ \dfrac{V_0 m}{k_B}\dfrac{n_H}{T_H} = \dfrac{n_H}{T_H K_0}\dfrac{2(d-1)}{d(d+2)}\, n k_B T \end{cases}$$

$$\Rightarrow = -2p\zeta_T^{(2)*}\Theta_{\underline{k}}(\tau) + \frac{p}{\cancel{2}}K^2\zeta_{T,1}^{(2)}\Theta_k(\tau)\frac{\cancel{2}(d-1)}{d(d+2)}\frac{n k_B T}{K_0} + \frac{p}{\cancel{2}}K^2\zeta_{T,2}^{(2)}\rho_{\underline{k}}(\tau)\frac{\cancel{2}(d-1)}{d(d+2)}\, n k_B T\, \frac{n_H}{T_H K_0}$$

⓷

$$\Rightarrow \left[\frac{\partial}{\partial\tau} + p\zeta_T^{(0)*} + \frac{d+2}{2(d-1)}\left(\kappa^* - p\zeta_{T,1}^*\right)K^2\right]\Theta_{\underline{k}}(\tau) + \left[2p\zeta_T^{(0)*} + \frac{(d+2)}{2(d-1)}\left(\mu^* - p\zeta_{T,2}^*\right)k^2\right]\rho_{\underline{k}}(\tau) + \frac{2}{d}\, i k\, w_{\underline{k}\,\parallel}(\tau) = 0$$